(I.C.)

152

AN

ANALYTICAL TREATISE

ON

PLANE AND SPHERICAL

TRIGONOMETRY,

AND

THE ANALYSIS OF ANGULAR SECTIONS.

SECOND EDITION,

CORRECTED AND IMPROVED.

DESIGNED FOR THE

USE OF STUDENTS IN THE UNIVERSITY OF LONDON.

BY THE

REV. DIONYSIUS LARDNER, LL.D., F.R.S.L. & E.

PROFESSOR OF NATURAL PHILOSOPHY AND ASTRONOMY IN THE UNIVERSITY OF LONDON, HON. F. P. S. CAM., M. R. I. A., F. AST. SOC. LON., &c. &c.

LONDON:

PRINTED FOR JOHN TAYLOR,

Bookseller and Publisher to the University of London,

30, GOWER-STREET;

AND SOLD BY JAMES DUNCAN, PATERNOSTER-ROW; HESSEY, FLEET-STREET; AND HATCHARD AND SON, PICCADILLY.

1828.

LONDON:
PRINTED BY THOMAS DAVISON, WHITEFRIARS.

TO

CHARLES BABBAGE, ESQ.

F.R.S.L. AND E., HON. M.R.I.A.,

&c. &c. &c.

AS

A SLIGHT TRIBUTE OF ADMIRATION FOR HIGH SCIENTIFIC TALENT

AND

ESTEEM FOR EMINENT PRIVATE WORTH,

THIS TREATISE

IS INSCRIBED

BY HIS FAITHFUL FRIEND,

THE AUTHOR.

PREFACE

TO

THE SECOND EDITION.

Trigonometry is a department of mathematical science which in its nature is essentially analytical. It borrows from geometry no principle except the proportionality of the sides of similar triangles, and even this property may perhaps be more simply and clearly expressed by the language of analysis than by the phraseology of geometry. All the results of this science are matters of *computation*, the quantities engaged in it are all *numerical*, and the operations to which these quantities are submitted are *arithmetical*. Some of the processes and quantities no doubt admit of geometrical expression; but in many instances this is not the case. Relations are contemplated and combinations are formed wholly foreign to geometry; problems are solved and investigations instituted under which the feeble,

though elegant, agency of that science would sink powerless. Nevertheless in most of the elementary works which have been published on this part of mathematics, and particularly those in our language, the subject has been presented under a geometrical form. An impression seems to have fixed itself on the minds of teachers that whatever was explained by geometry was easily comprehended, but that the attainment of any thing which assumed an algebraical form was necessarily attended with much difficulty. The fact that geometrical reasoning was understood with more facility than the language and principles of algebra, is not to be questioned. I apprehend, however, that this did not arise from the difficulty of the algebraical method or the superior simplicity of the geometrical, but from the circumstance of the student's having previously obtained an extensive and accurate acquaintance with geometry and a practical familiarity with its processes, while algebra was either wholly omitted or comparatively neglected. Let a student be equally familiar with geometry and algebra, and it will soon appear how much greater facility he will acquire in trigonometrical investigations by the latter.

The system has however been changed, and analytical science has in these countries at length obtained that attention as an elementary part of mathematical education, to which its importance so justly entitles it. Students who are about to commence trigonometry have now generally obtained a competent knowledge of algebra, and the reasons which hitherto rendered it expedient to treat the subject geometrically no longer exist. In the following treatise I have accordingly brought to my aid the powerful resources of analysis. On the property of similar triangles, already mentioned, as a basis, I have attempted to raise the whole superstructure of trigonometrical science by reasoning purely analytical. Nor have I found it necessary to resort to any principles beyond what must be considered the rudiments of algebra, except in those higher departments of trigonometry which are only read by students who have made considerable progress in mathematics. Those who are conversant with the first principles of elementary algebra are competent to study all those parts of the present work which are necessary for the elements of natural philosophy, and which are distinguished by an asterisk in the table of contents.

The power and facility of investigation which the student obtains by the use of the analytical method are not its only advantages. The great generality of the theorems, the beautiful symmetry which reigns among the groups of results, the order with which they are developed one from another, offering themselves as unavoidable consequences of the method, and almost independent of the will or the skill of the author, the singular fitness with which the symbolical language of analysis adapts itself so as to represent, even to the eye, all this order and harmony, are effects too conspicuous not to be immediately noticed. Nor is the elegant form which the science thus receives from the hand of analysis a mere object pleasurable to contemplate, but barren of utility. All this order and symmetry, which is given as well to the matter as the form, as well to the things expressed as to the characters which express them, not only serves to impress the knowledge indelibly on the memory, but is the fruitful source of further improvement and discovery.

The table of contents presents so complete an analysis of the work, that any further account of its arrangement would be superfluous. The first

three sections of the second part might, perhaps, more properly come under the title of spherical geometry. However, as the formulæ and theorems of spherical trigonometry have an intimate and necessary connexion with the subject of these sections, and as they are not contained in other works commonly used in the universities, to have omitted them would be going too far in the sacrifice of utility to system. I have devoted considerable attention to the section on the solution of spherical triangles, and hope that I have succeeded in rendering the discussion of it more full and satisfactory than is usual.

Geodæsy, a subject of peculiar interest and much neglected in trigonometrical works, occupies the tenth section of the second part. This would form an interesting subject for a separate treatise; but as we have no such work in our language, I conceived that it might be useful to introduce in the present work the rapid sketch contained in that section. For the materials of it I am indebted for the most part to *Base du Systême Metrique*, &c. of *Delambre*, *Traité de Géodésie* of *Puissant*, *The Survey of England and Wales* by *General*

Roy, Col. Mudge, &c. besides several other sources of later information.

To *Poinsot's* Memoir on the Analysis of Angular Sections I owe those important corrections and improvements of the common formulæ, expressing the relations of multiple arcs, which are now for the first time introduced into an elementary treatise.

In this second edition numerous corrections and improvements have been introduced in almost every part of the work. Many of the most important have been suggested by my friend Professor De Morgan; among which the demonstration of the fundamental formulæ in p. 28 merits particular notice. The demonstration given in the former edition was derived from the property of a quadrilateral inscribed in a circle, of which, indeed, the formulæ themselves are little more than an analytical translation. The present proof, however, has the advantage of being derived immediately from the definitions of the sine and cosine previously given in p. 16. By this change all the results of the science are, as they ought to be, connected immediately with its definitions.

The articles marked with an asterisk in the

Table of Contents form the course which Professor De Morgan recommends for those students in the university who intend to limit their mathematical studies in the university to one year.

Extensive tables of trigonometrical formulæ are placed at the end of the volume for the general purposes of reference.

London,
33, *Percy-street, Bedford-square,*
May, 1828.

CORRIGENDA.

Page 96, line ult. *for* sin.c —, *read* sin.c +.

120, line 2 from the bottom, *for* cos.a =, *read* cos.A =.

Table IV. 60, *for* $\tan.^2(45^\circ + \frac{1}{2}\omega)$, *read* $\tan.^2(45^\circ - \frac{1}{2}\omega)$.

CONTENTS.

PART I.

THE ELEMENTS OF PLANE TRIGONOMETRY.

* SECTION I.

* SECTION II.

SECTION III.

* SECTION IV.

* SECTION V.

I. *The solution of right-angled triangles.*

II. *The solution of oblique-angled triangles.*

Note on logarithms.

SECTION VI.

PART II.

SPHERICAL TRIGONOMETRY.

* SECTION I.

*SECTION II.

* SECTION III.

SECTION VI.

SECTION VII.

SECTION VIII.

SECTION IX.

SECTION X.

SECTION IV.

SECTION V.

SECTION VI.

AN

ANALYTICAL TREATISE

ON

TRIGONOMETRY.

PART I.

THE ELEMENTS OF PLANE TRIGONOMETRY.

SECTION I.

Of Angles and Arcs.

(1.) THE object of Trigonometry was originally the solution of problems, in which some of the sides and angles of a triangle are given to compute the others. Like all parts of science, however, its objects became more extensive as knowledge advanced and discovery accumulated; and this department of mathematical science, which was at first confined to the solution of one general problem, has now spread its uses over the whole of the immense domains of the mathematical and physical sciences. In the wide range of modern analysis, there is scarcely a subject of investigation to which trigonometry has not imparted clearness and perspicuity by the use of its language and its principles; and the physical investigations of philosophers of our times are still more

largely indebted for their conciseness, elegance, and generality, to the symbols and established formulæ of this science.

In its present improved and enlarged state, trigonometry might not improperly be called the *angular calculus;* for however extensive and various its more remote uses and applications may be, its immediate object is to institute a system of symbols, and to establish principles by which angular magnitude may be submitted to computation, and numerically connected with other species of magnitude; so that angles and the quantities on which they depend, or which depend on them, may be united in the same analytical formulæ, and may have their mutual relations investigated by the same methods of computation that are applied to all other quantities.

(2.) There are two methods by which angular magnitude may be numerically expressed. The first consists in assuming arbitrarily some angle as the angular unit, and expressing other angles by the numbers which are related to the numeral unit in the same manner as the proposed angles are to the angular unit.

Two right lines drawn through the same point at right angles divide the space surrounding the point and in the plane of the lines, into four equal angular spaces called right angles. If lines be supposed to be drawn through the point of intersection of these two lines dividing each of the four right angles into ninety equal angles, each of these angles is called a *degree;* and therefore the entire angular space around the point consists of four times ninety, or 360 degrees.

If each of these angles, called *degrees,* be divided into 60 equal angles, each of these smaller subdivisions is called *a minute.*

In like manner each *minute* being subdivided into sixty equal angles, these subdivisions are called *seconds.*

A second is the smallest angle which has received a distinct denomination. All smaller angles are usually expressed as decimal parts of a second.

Thus, according to this division of angular magnitude, a second may be considered as the angular unit.

In astronomy another denomination is sometimes used; the third part of a right angle, or thirty degrees, being called a *sign*.

Thus, all the angular space in the same plane surrounding a point consists of twelve *signs* *.

Degrees are expressed by 0 placed over their number.

Thus, forty-five degrees, or half a right angle, is expressed 45^0; thirty degrees, or a third of a right angle, thus, 30^0.

Minutes are expressed by an accent $'$ placed over their number, thus, $5'$ signifies five minutes; and seconds by a double accent $''$, thus, $5''$ signifies five seconds.

Thus,

$$35^0 \quad 17' \quad 10''.5$$

signifies thirty-five degrees + 17 minutes + 10 seconds + 5 tenths of a second.

In a similar way *s* over the number denotes *signs*. Hence the meaning of

$$4^s \quad 25^0 \quad 17' \quad 10''.5$$

is obvious.

(3.) In some foreign mathematical and physical works a different division of angles is used, with which it is necessary the student should be acquainted. It has been long considered that the division of the right angle into ninety equal

---

* This division of the angular space round a point arose from the twelve signs of the zodiac, each of which occupies thirty degrees.

parts was unnatural and inconvenient, and several mathematicians, both British and continental, have from time to time proposed a decimal division. This has been actually carried into effect in France, and adopted by many writers of that country. They divide the angular space round a point into four hundred equal parts, which they call degrees.

Each degree is divided into a hundred minutes, and each minute into a hundred seconds, and so on.

Thus it is equally easy to express an angle in degrees and decimal parts of a degree, as in degrees, minutes, and seconds,

$$36^{\circ}.567329 = 36^{\circ} \quad 56' \quad 73'' \quad 29'''.$$

The mark $'''$ denoting hundredth parts of a second.

The degree may here, therefore, be taken as the angular unit.

The former division is called the *sexagesimal*, and the latter the *decimal* division.

(4.) The sexagesimal division is generally thought to have originated with the Egyptians, who supposed the year to consist of 360 days, and, therefore, that the sun described in each day the 360th part of four right angles. Besides, it has been considered that the number 360 is convenient in admitting of a great number of divisors, as 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, &c.

Whatever be the advantages of these different divisions, custom has almost universally prevailed in favour of the sexagesimal division. In the present treatise, therefore, we shall generally use it. It may be useful, however, to give a general rule for translating the expressions for angles according to either division into expressions for the same angles according to the other.

Let x be a degree of the decimal division, and y a degree of the sexagesimal division. Hence

$$100x = 90y, \text{ or } 10x = 9y,$$
$$\because x = y \times 0.9,$$
$$y = x + \frac{x}{9}.$$

Again, let x be a minute of the decimal, and y of the sexagesimal division. Hence $100x$ is a decimal degree, and $60y$ a sexagesimal degree.

Substituting these for x and y in the former equation, we find

$$100x = 60y \times 0.9,$$
$$\because x = 6y \times 0.09;$$
$$\text{or } x = \frac{27}{50}y,$$
$$\because y = \frac{50}{27}x.$$

In like manner, if x be a decimal second, and y a sexagesimal second, by substituting $100x$ and $60y$ for x and y in the last equation, we obtain

$$x = \frac{3^4}{250}y,$$
$$\because y = \frac{250}{3^4}x.$$

By which formulæ, angles expressed relatively to either division may be translated into the other.

In transmuting decimal into sexagesimal denominations, the following table may be useful:

Decimal.	Sexagesimal.	Sexagesimal.	Decimal.
1°	$= 54' = 3240''$	1°	$= 1^\circ\ 11'\ 11''\ 11'''$, &c.
$1'$	$= 32''.4$	$1'$	$= 1'\ 85''\ 18'''.51$, &c.
$1''$	$= 0''.324.$	$1''$	$= 3''\ 8'''.64.$

From this table the value of any number of decimal degrees, minutes, and seconds, may be obtained in sexagesimal degrees, minutes, and seconds, by simple multiplication.

(5.) The angular space surrounding a point being sup-

posed to be divided by lines intersecting at that point into any number of angles of any magnitudes, if a circle be described with any line as radius, and that point as centre, the circumference of that circle will be divided into as many arcs as the angular space surrounding the point is divided into angles, and the relation of the magnitudes of these arcs is the same as that of the angles, each arc bearing to the entire circumference of the circle the same ratio as the corresponding angle bears to four right angles.

It is usual to give an arc of a circle the same denomination as that of the angle which it subtends at the centre of the circle. Thus, arcs of a circle which subtend at the centre angles of 1°, 2°, or 3°, are called one, two, or three degrees of the circumference of a circle, and the same observation extends to minutes, seconds, &c.

These denominations, however, when used to express circular arcs, do not express their absolute length, but only denote their ratios to the whole circumference; thus an arc of one degree is the 360th part of the entire circumference.

(6.) Similar arcs of circles are defined in geometry to be those which bear the same ratio to their respective circumferences, from whence it follows that they must consist of the same number of degrees, minutes, and seconds, or be arcs of the same denomination. Let n be the denomination of two similar arcs A, A′, of which the circumferences are C, C′. Hence

$$\text{A} = \frac{n}{360}\text{C}, \ \text{A}' = \frac{n}{360}\text{C}';$$

these formulæ show the absolute length of the arc when its denomination and the absolute length of the whole circumference are known.

It is an established principle of geometry, that the circumferences of different circles are proportional to their radii; and hence we infer that similar arcs of circles are

also proportional to their radii, and *vice versa*. Two arcs of different circles, therefore, which bear the same ratio to their respective radii must be similar, and therefore consist of the same number of degrees, minutes, and seconds.

(7.) From these principles it follows that an arc of one second of all circles is contained the same number of times in their radii, and from the calculation of the ratio of the circumference of a circle to its diameter [Geometry (375.)], it is known that this number differs from 206265 by a small fraction. Therefore the radius of any circle differs from an arc of 206265 seconds by a small fractional part of a second. The circumference of a circle being incommensurable with its diameter, it is impossible to express the exact length of the radius in seconds and parts of a second.

The number above mentioned, however, gives its length with sufficient accuracy for practical purposes.

We are thus enabled to solve the following problems.

PROP. I.

(8.) *Given the radius of a circle and an angle in seconds at its centre, to compute the length of the corresponding arc of the circle.*

Let r be the radius, n the angle in seconds, and x the arc.

By the principles just laid down, it appears that

$$206265 : n :: r : x = r \cdot \frac{n}{206265}.$$

Thus, if the distance (r) of any object, as the sun, moon, or a planet, be given, and its apparent magnitude (n), or the angle it subtends at the eye, be measured, its actual diameter (x) may be computed.

PROP. II.

(9.) *Given the length of a circular arc and the angle in seconds which it subtends at the centre of the circle, to compute the length of the radius.*

Let a be the arc, n the seconds of the angle, and x the radius. Hence

$$n : 206265 :: a : x = a \cdot \frac{206265}{n}.$$

Thus, if the absolute diameter (a) of any object, as the sun, moon, or a planet, be given, and its apparent diameter (n) be measured, its distance (x) may be computed.

PROP. III.

(10.) *Given the length of a circular arc and the length of its radius, to compute the seconds in the angle which it subtends at the centre of the circle.*

Let a be the arc, r the radius, and x the seconds in the angle. Hence

$$r : a :: 206265 : x = 206265 \cdot \frac{a}{r}.$$

(11.) *Cor.* Hence, angles in general are proportional to their arcs divided by their radii, or they are directly as their arcs, and inversely as their radii.

(12.) The method of expressing angles and arcs explained in the preceding articles refers all angles to an angular unit of a degree, minute, or second; and all arcs, as well as their radii, are computed by the number of seconds of the circumference which they contain in their length. There is, however, another method of expressing these quantities,

which is used in works of science as frequently as the former, and in which they are all referred to a radius which is considered as unity.

The exponent of the ratio of the circumference of a circle to its diameter being expressed in general by the symbol π, the numerical value of π determined by the differential calculus is [Geom. (375.)],

$$\pi = 3.141592.$$

This number then expresses, *quam proxime*, the ratio of the semicircumference of a circle to its radius, so that if r be the radius, and c the circumference, we have

$$c = 2r\pi.$$

By means of this number π we obtain another method of expressing angles by the numerical exponent of the ratio of the arcs which they subtend to their radii. Thus, if ω in this way express an angle, it is meant that ω is a number which bears to unity the ratio which the arc subtending the proposed angle bears to its radius. This number must be evidently the same for all arcs which subtend the same angle, since, being similar, they bear the same ratio to their respective radii. Angles thus expressed may be considered equivalent to arcs whose radius is unity.

PROP. IV.

(13.) *Given the radius of a circle and an angle at its centre related to the radius unity, to compute the length of the corresponding arc.*

Let the radius be r, the angle ω, and the arc x. Hence

$$1 : \omega :: r : x,$$

$$\therefore x = r\omega.$$

PROP. V.

(14.) *Given the length of an arc and the angle which it subtends at the centre related to the radius unity, to compute the length of the radius.*

Let a be the arc, ω the angle, and x the radius.

As before,

$$x = \frac{a}{\omega}.$$

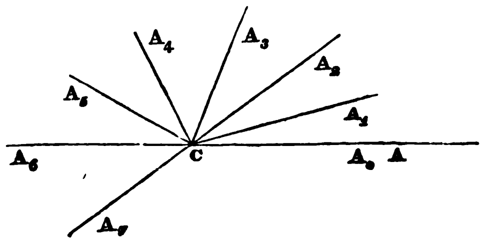
PROP. VI.

(15.) *Given the length of an arc and the length of its radius, to compute the angle which it subtends at the centre related to the radius unity.*

Let a be the arc, r the radius, and ω the angle,

$$\omega = \frac{a}{r}.$$

(16.) Angles contemplated in common geometry being in general the angles of triangles, do not exceed two right angles, or 180°, as explained in (2.), or π as in (12.). Some geometers, however, considering polygons as susceptible of *reentrant* angles, have extended their views to angles greater than 180°, or π, but less than 360°, or 2π. We shall, however, take a more extensive view of angular magnitude, and consider it, like all other species of quantity, as susceptible of unlimited increase, as well as unlimited diminution. Let any point C be assumed, and a line CA drawn from that point in any direction CA be considered fixed. Through the same point C let another indefinite right line CA_0 be conceived to be drawn and moved round the point C in the



same plane, commencing from coincidence with the fixed line CA, and moving in the direction A_1, A_2, A_3, A_4 . . . . . . The angular space which this line has described in its revolution from its coincidence with CA being ω, it is not difficult to conceive ω susceptible of unlimited increase. When the revolving line assumes the position CA_6 *in directum* with CA_0 its first position, it has described an angle $= \pi$, or 180°. When below the position CA_6 as at CA_7, the angle ω is greater than π, and increases until it again coincides with CA when $\omega = 2\pi$. It then commences a second revolution, when the angular space it has described is greater than 2π, and by continuing to revolve it successively becomes 3π, 4π, 5π, ... every even multiple of π corresponding to a return to its primitive position, and every odd multiple marking exactly the opposite direction. Thus angular magnitude may be conceived to be susceptible of unlimited increase.

(17.) To explain the method of expressing angles of different degrees of magnitude, let two indefinite right lines be assumed intersecting at a fixed point C at right angles, and whose position we shall consider fixed. In their plane let an indefinite right line be conceived to revolve round the centre C as already explained, and in its initial position let it coincide with CA. Let ω express, in general, an angle which is less than a right angle.

The angular motion of the revolving line originating in CA, it will be found successively in the positions Ca, CA$'$, Ca', (P 18

after describing one, two, and three right angles, that is, after describing the angles $\frac{\pi}{2}$, π, and $\frac{3\pi}{2}$ respectively. It will be found in its initial position after describing four right angles or 2π.

In the second revolution, when it attains these four positions, it has described 5, 6, 7, and 8, right angles, or $\frac{5\pi}{2}$, 3π, $\frac{7\pi}{2}$, and 4π.

In general, when the revolving line assumes the position CA, it has described some complete number of revolutions, and as one revolution is expressed by 2π, the angle it has described, whenever it assumes this position, must be expressed by $2n\pi$, n denoting any term of the series

$$0, 1, 2, 3, \ldots$$

To express, in general, the angle which it has described when it assumes the positions Ca, CA$'$, and Ca', it is only necessary to add one, two, and three right angles, or $\frac{\pi}{2}$, π, and $\frac{3\pi}{2}$, to the angle $2n\pi$, which it has described when it assumes the position Ca. Hence, at Ca it has described

$$(2n + \tfrac{1}{2})\pi.$$

At CA$'$,

$$(2n + 1)\pi;$$

and at Ca',

$$(2n + \tfrac{3}{2})\pi,$$

as indicated in Table I.

The angle which the revolving line has described when it is in the angle ACa may be expressed either by adding ω to the angle described in the position CA, or subtracting ω from that described in the position Ca. Hence it is expressed by either of the formulæ

$$2n\pi + \omega,$$
$$(2n + \tfrac{1}{2})\pi - \omega.$$

In like manner the angle described when it is in the angle aCA$'$ may be expressed either by adding ω to the angle described in the position Ca, or subtracting ω from that described in the position CA$'$. Hence it is expressed by either of the formulæ

$$(2n + \tfrac{1}{2})\pi + \omega,$$
$$(2n + 1)\pi - \omega.$$

By methods exactly similar we may express the angle described when the revolving line is in the angle A$'$Ca' by either of the formulæ

$$(2n + 1)\pi + \omega,$$
$$(2n + \tfrac{3}{2})\pi - \omega.$$

And the angle described when it is in the angle a'CA by either of the formulæ

$$(2n + \tfrac{3}{2})\pi + \omega,$$
$$2n\pi - \omega.$$

These results are all indicated in Tab. I. (end of book)

(18.) A negative angle $-\omega$ should be measured from CA towards Ca' in the direction opposite to that of the positive angles. For $-\omega$ is what $2n\pi - \omega$ becomes when $n = 0$, and this in general indicates an angle in ACa' the revolving line making with CA the angle ω.

Circular arcs described round the centre C with a radius equal to the linear unit may be obviously expressed in the same manner as has been already applied to the angles; and if the radius be r, the arcs will be found by multiplying r (13.) by the expressions already found for the angles; thus, ω becomes $r\omega$; $2n\pi + \omega$ becomes $r(2n\pi + \omega)$, &c. &c.

(19.) The angle $\frac{\pi}{2} - \omega$ is called the *complement* of ω, and the angle $\pi - \omega$ is called the *supplement* of ω.

SECTION II.

Definitions of trigonometrical terms and their mutual relations.

(20.) It is an established property of a right angled triangle, that if the ratio of any pair of its sides be known, the angles and the ratios of the other sides may be found. This forms the fundamental principle of trigonometry, the exponents of these ratios being here adopted as the criterions for the determination of the angles.

To illustrate this, suppose that it be given, that the number $\frac{2}{3}$ is the exponent of the ratio of the side of a right angled triangle to its hypothenuse, and it is required to determine the opposite angle. Upon any right line as diameter, let a semicircle be described, and from either extremity let a chord be inflected equal to two thirds of the diameter, and the triangle completed. The angle opposite this chord is the sought angle.

In place, however, of effecting this construction, the computist has only to refer to tables which have already been calculated, in which the magnitude of the angle he seeks is registered with the given exponent $\frac{2}{3}$, which determines it. Of the methods for constructing such tables we shall speak hereafter.

(21.) As there are three pairs of sides in a right angled triangle differently related to either of its acute angles, so there are three ratios which will determine the angle.

Let ω be the angle, and y the opposite side, x the containing side, and r the hypothenuse; the angle ω may be indifferently determined by any of the three numbers

$$\frac{y}{r}, \quad \frac{y}{x}, \quad \frac{r}{x}.$$

The first $\frac{y}{r}$ is called the *sine* of the angle ω, the second $\frac{y}{x}$ is called its *tangent*, and the third $\frac{r}{x}$ is called its *secant*. The origin of these denominations we shall presently explain.

The three ratios which are the reciprocals of those already expressed, scil.

$$\frac{x}{r}, \quad \frac{x}{y}, \quad \frac{r}{y},$$

bear the same relation to the other acute angle of the triangle as the former do to the assumed one. One acute angle being the *complement* (19.) of the other, it follows that $\frac{x}{r}$, $\frac{x}{y}$, and $\frac{r}{y}$, are the sine, tangent, and secant of the complement of the proposed angle, and are thence called its *co*-sine, *co*-tangent, and *co*-secant.

(22.) It appears from what has been just stated that these trigonometrical quantities, sines, tangents, secants, cosines, co-tangents, and co-secants, are properly *numbers*, which are the exponents of the ratios of the lengths of certain right lines by which the angles may always be constructed, or which, by reference to computed tables, will immediately indicate them. It is usual, however, to represent these quantities by right lines which are proportional to them, a certain length being first arbitrarily assumed as the linear unit.

With the centre C, and the linear unit CA as radius, let a circle be described, and let another radius CA″ be drawn, making any angle ω with the initial radius CA. From A″ draw the perpendicular A″P to the initial radius CA, and from A draw the tangent, and produce CA″ to meet it at T. Now, if y, x, and r, be the three sides of a right

angled triangle similar to A''PC, ω being the angle opposite to y, we have

$$\frac{y}{r} = \frac{\text{A}''\text{P}}{\text{CA}} = \sin.\ \omega.$$

But since CA is assumed as the linear unit, $\therefore$ A''P $= \sin.\ \omega$.

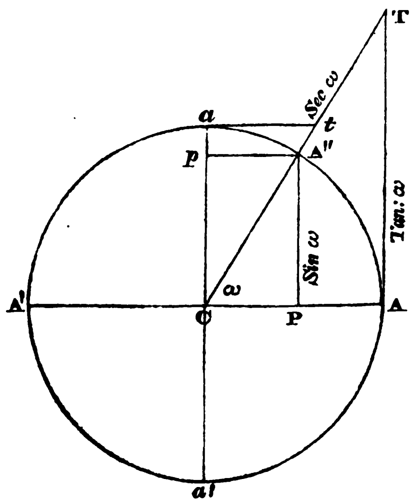
Again, since TCA is also similar to the same triangle, we have

$$\frac{y}{x} = \frac{\text{TA}}{\text{CA}} = \tan.\ \omega,$$

$$\therefore \text{TA} = \tan.\ \omega,$$

$$\frac{r}{x} = \frac{\text{CT}}{\text{CA}} = \sec.\ \omega,$$

$$\therefore \text{CT} = \sec.\ \omega.$$



(23.) Besides these terms, there are three others sometimes, though not so frequently, used. AP is called the *versed sine* of ω, *ap* its *co*-versed sine being the versed sine of its *complement*, and A'P its *su*-versed sine being the versed sine of its *supplement*. These, like the other quantities,

are generally expressed by the numbers which denote their ratios to the radius.

The line joining the extremities of an arc is called its *chord*.

The cause of the denominations, tangent, and secant, are hence obvious.

It is evident that $A''p = CP$ is the cosine of ω, or sine of the complement, and that *at* and *ct* are its cotangent and cosecant.

(24.) The radius of the circle, to which, as a fixed quantity, the sines, &c. are related, being arbitrary, may, under different circumstances, be assumed with different values; and hence it will happen that the sine, cosine, &c. of the same angle will have different values relatively to different radii. Since, however, these quantities, sine, cosine, &c. are homologous sides of similar right angled triangles, they will always, for the same angle, be proportional to each other, and to the fixed quantities or radii to which they are all related. Thus, if s and s′ be the sines, or cosines, or tangents, &c. of the angle ω, the first related to the radius R, and the second to R′, we have

$$s : s' :: R : R'.$$

If s be related to unity as radius, we have

$$s = \frac{s'}{R'}.$$

This furnishes a rule by which any formula composed of trigonometrical quantities related to the radius unity may be transformed into another related to any proposed radius R′. "*Let each sine, cosine, &c. which is involved in the formula related to the radius unity be divided by the proposed radius, and the formula will hold good as related to this new radius.*"

Thus, if the formula

$$\sin.^2\varphi + \cos.^2\varphi = 1$$

be established relatively to the radius unity, a similar formula related to the radius R is

$$\frac{\sin.^2\phi}{R^2} + \frac{\cos.^2\phi}{R^2} = 1,$$

$$\text{or } \sin.^2\phi + \cos.^2\phi = R^2.$$

Thus, if, after dividing the different terms of the formula by the proper dimensions of R, the whole be multiplied by the highest power of R which occurs in the denominators, the whole formula will become *homogeneous* with respect to the several quantities, the sines, cosines, &c. and R, that is to say, each term will contain the same number of simple factors of these quantities, or will be of the same *dimensions*.

Hence the rule already given may thus be modified:

"*Let the whole formula related to the radius unity be rendered homogeneous by introducing as a factor into each term such a power of the radius as will render the dimensions of that term equal to those of the term of the highest dimensions in the proposed formula.*"

Thus, the formula

$$\cos.2\phi = 2\cos.^2\phi - 1$$

would become

$$R\cos.2\phi = 2\cos.^2\phi - R^2,$$

which is immediately deduced from

$$\frac{\cos.2\phi}{R} = 2\frac{\cos.^2\phi}{R^2} - 1,$$

obtained by the first rule.

Having established this rule, we shall in the succeeding part of this work always refer the trigonometrical quantities to the radius unity, making the necessary transformation whenever the introduction of any other radius becomes necessary.

(25.) From the definitions of the several trigonometrical

terms their mutual relations become manifest. By the definitions, we have

$$\sin.\omega = \frac{y}{r} \text{ [1]}, \quad \tan.\omega = \frac{y}{x} \text{ [2]}, \quad \sec.\omega = \frac{r}{x} \text{ [3]},$$

$$\cos.\omega = \frac{x}{r} \text{ [4]}, \quad \cot.\omega = \frac{x}{y} \text{ [5]}, \quad \text{cosec}.\omega = \frac{r}{y} \text{ [6]}.$$

(26.) By squaring [1] and [4], and adding the results, we have

$$\sin.^2\omega + \cos.^2\omega = 1,$$

since $r^2 = x^2 + y^2$.

(27.) By dividing [1] by [4], we have

$$\frac{\sin.\omega}{\cos.\omega} = \frac{y}{x}, \quad \because \frac{\sin.\omega}{\cos.\omega} = \tan.\omega.$$

(28.) By multiplying [2] and [5], we obtain

$$\tan.\omega \cot.\omega = 1.$$

Thus the tangent and cotangent are reciprocals.

(29.) In like manner, by multiplying [3] and [4], we have

$$\sec.\omega \cos.\omega = 1.$$

The secant and cosine are therefore reciprocals.

(30.) It follows also by multiplying [1] and [6], that

$$\text{cosec}.\omega \sin.\omega = 1,$$

therefore the cosecant and sine are reciprocals.

(31.) By squaring [2], and adding unity to the result, we find

$$\tan.^2\omega + 1 = \frac{y^2 + x^2}{x^2} = \frac{r^2}{x^2},$$

$$\because \sec.^2\omega = 1 + \tan.^2\omega.$$

(32.) In like manner, by squaring [5] and adding unity, we obtain

$$\text{cosec}.^2\omega = 1 + \cot.^2\omega.$$

(33.) By (27.) and (28.) it follows that

$$\frac{\cos.\omega}{\sin.\omega} = \cot.\omega.$$

(34.) By the definitions (23.) of the versed sine, covered sine, and suversed sine, we have

$$\text{ver. sin.}\omega = 1 - \cos.\omega,$$
$$\text{cov. sin.}\omega = 1 - \sin.\omega,$$
$$\text{suv. sin.}\omega = 1 + \cos.\omega.$$

(35.) The preceding results, which are of considerable importance in all trigonometrical investigations, are collected in the following table. By the relations here given, any one of the trigonometrical terms may be expressed in terms of any of the others. This gives a variety of problems which are solved by mere elimination by the equations of this table.

TABLE II.

1.	$\sin.^2\omega + \cos.^2\omega = 1.$
2.	$\dfrac{\sin.\omega}{\cos.\omega} = \tan.\omega.$
3.	$\dfrac{\cos.\omega}{\sin.\omega} = \cot.\omega.$
4.	$\tan.\omega \cot.\omega = 1.$
5.	$\sec.\omega \cos.\omega = 1.$
6.	$\text{cosec.}\omega \sin.\omega = 1.$
7.	$1 + \tan.^2\omega = \sec.^2\omega.$
8.	$1 + \cot.^2\omega = \text{cosec.}^2\omega.$
9.	$\text{ver. sin.}\omega = 1 - \cos.\omega.$
10.	$\text{cov. sin.}\omega = 1 - \sin.\omega.$
11.	$\text{suv. sin.}\omega = 1 + \cos.\omega.$

By these equations, any one of the quantities, $\sin.\omega$, $\cos.\omega$, &c. being given, all the others may be determined.

These investigations will furnish a useful exercise for the student.

(36.) The trigonometrical quantities, like all other alge-

braical quantities, are susceptible of different signs under different circumstances. The signs of the sine and cosine are determined by the same rules as those of the co-ordinates of a point in analytic geometry. Mathematicians have differed as to the principles by which the signs of these quantities should be explained, some determining them by the principle that every quantity must change its sign in passing through zero; and others, that lines placed in directions immediately opposite should be characterised by algebraical species with opposite signs; others again considering the regulation of signs purely conventional.

To determine the variation of the signs of the sine and cosine, let C be the centre of a circle whose radius CA is unity, and, as before, let CA be the initial position of the revolving radius.

When the radius coincides with CA, the sine of the arc $= 0$, and the cosine $= 1$. This will be manifest from considering the triangle CPM to be continually changed by the radius CP approaching CA. During this change PM, the sine of the angle, continually diminishes, and CM its cosine continually approaches to equality with CA or unity; and when the angle at C actually vanishes, and CP coincides with CA, then PM vanishes, and CM becomes equal to CA.

Hence for all angles terminated by the radius CA, the sine $= 0$, and the cosine $= 1$.

While the revolving radius moves from CA through the angle ACa, the sine PM continually increases, and the cosine decreases until it coincides with Ca, where the sine coincides with Ca, and is $\because = 1$, and the cosine vanishes, or $= 0$.

Through the angle ACa we shall consider the sine and cosine, both positive, and, in general, we shall consider those sines which are measured from the diameter AA$'$ in the direction Ca as positive, and those which are measured in the opposite direction Ca' as negative.

Also, those cosines which are measured in the direction CA we shall consider positive, and those in the opposite direction CA′ negative. It will be found that by this arrangement, all trigonometrical quantities will change their signs upon passing through zero and infinity. As the radius revolves through the angle aCA′ from Ca towards CA′, the sine diminishes, but is still positive, and the cosine increases, and is negative. Thus, at Ca the cosine passes through zero, and changes its sign. When the radius coincides with CA′, the sine $= 0$, and the cosine coinciding with CA′ is $= -1$.

Through the angle A′Ca' the sine increases, and the cosine diminishes, both being negative, the sine changing its sign in passing through 0 at CA′; and at Ca' the cosine vanishes, and the sine coincides with CA′, and $\because = -1$.

Through a'CA the sine diminishes continuing negative, and the cosine increases and is positive, thus changing its sign in passing through zero at Ca'.

Thus the several changes which the sine and cosine undergo in one revolution of the radius are evident, and they suffer the same changes every revolution.

(37.) By the formulæ 2 and 3 of Tab. II., it follows that the tangent and cotangent are positive when the sine and cosine have like signs, and negative when they have unlike signs; and by 5, it appears that the sign of the secant is always that of the cosine; and by 6, that the sign of the cosecant is always that of the sine. Thus the signs of the sine and cosine regulate those of all other trigonometrical terms.

By (17.), it appears that all angles terminated at CA are in general expressed by $2n\pi$. By what we have just established, we have

$$\sin.2n\pi = 0, \qquad \cos.2n\pi = +1;$$

and by 2, 3, 5, 6, of Tab. II., we infer that

$$\tan.2n\pi = 0, \qquad \cot.2n\pi = \infty,$$
$$\sec.2n\pi = +1, \qquad \operatorname{cosec}.2n\pi = \infty.$$

Since the sine and cosine are both positive in the angle ACa, it follows by what has been already proved, that the tangent, cotangent, secant, and cosecant, of any angle expressed by (17.),

$$2n\pi + \omega,$$

ω being $< \frac{\pi}{2}$; must be all positive.

The angles terminated in Ca are expressed by $(2n + \frac{1}{2})\pi$, and we have, as before,

$$\sin.(2n + \tfrac{1}{2})\pi = +1, \qquad \cos.(2n + \tfrac{1}{2})\pi = 0,$$
$$\because \tan.(2n + \tfrac{1}{2})\pi = +\infty, \qquad \cot.(2n + \tfrac{1}{2})\pi = 0,$$
$$\sec.(2n + \tfrac{1}{2})\pi = \infty, \qquad \operatorname{cosec}.(2n + \tfrac{1}{2})\pi = +1.$$

An angle terminated in aCA$'$ is in general expressed by $(2n + 1)\pi - \omega$, and we have already shown that its sine is positive, and its cosine negative. Hence it follows by 2, 3, Tab. II., that its tangent and cotangent are negative, and by 5, that its secant is negative, and by 6, that its cosecant is positive.

An angle terminated at CA$'$ is in general expressed by $(2n + 1)\pi$, and by similar reasoning we conclude that

$$\sin.(2n + 1)\pi = 0, \qquad \cos.(2n + 1)\pi = -1,$$
$$\tan.(2n + 1)\pi = 0, \qquad \cot.(2n + 1)\pi = -\infty,$$
$$\sec.(2n + 1)\pi = -1, \qquad \operatorname{cosec}.(2n + 1)\pi = \infty.$$

The angles terminated in A$'$Ca' expressed by $(2n+1)\pi+\omega$, having their sines and cosines both negative, must have their tangents and cotangents both positive, and their secants and cosecants both negative by the same reasoning.

The angles terminated in Ca' are expressed by $(2n + \frac{3}{2})\pi$, and we have

$$\sin.(2n + \tfrac{3}{2})\pi = -1, \qquad \cos.(2n + \tfrac{3}{2})\pi = 0,$$
$$\tan.(2n + \tfrac{3}{2})\pi = -\infty, \qquad \cot.(2n + \tfrac{3}{2})\pi = 0,$$
$$\sec.(2n + \tfrac{3}{2})\pi = \infty, \qquad \operatorname{cosec}.(2n + \tfrac{3}{2})\pi = -1.$$

The angles terminated in a'CA, and expressed by $2n\pi - \omega$, having their signs negative, and their cosines positive, must have their tangents and cotangents negative, their secants positive, and their cosecants negative.

These results are collected and presented in one view in Table III.

(38.) By the diagram of this table, the following results are also obvious.

$$\begin{aligned}
&\cos.(2n\pi + \omega) = \cos.(2n\pi - \omega),\\
&\cos.[(2n + 1)\pi - \omega] = \cos.[(2n + 1)\pi + \omega],\\
&\cos.(2n\pi \pm \omega) = -\cos.[(2n + 1)\pi \pm \omega],\\
&\sin.(2n\pi + \omega) = \sin.[(2n + 1)\pi - \omega],\\
&\sin.(2n\pi - \omega) = \sin.[(2n + 1)\pi + \omega],\\
&\sin.(2n\pi + \omega) = -\sin.(2n\pi - \omega),\\
&\sin.[(2n + 1)\pi - \omega] = -\sin.[(2n + 1)\pi + \omega],\\
&\sin.(2n\pi \pm \omega) = -\sin.[(2n + 1)\pi \pm \omega].
\end{aligned}$$

(39.) In the first and sixth of these formulæ, if $n = 0$, we have

$$\begin{aligned}
&\cos.(+\omega) = \cos.(-\omega),\\
&\sin.(+\omega) = -\sin.(-\omega),
\end{aligned}$$

by which it appears that a change in the sign of an angle makes no change in its cosine, but changes the sign of its sine.

(40.) It is not difficult to perceive that the sine of an arc terminated at $a \pm \omega$ is equal to the cosine of an arc terminated at A $\pm\ \omega$, and has the same sign, and differs only in sign from the cosine of an arc terminated at A$' \pm \omega$. Also, that the cosine of an arc terminated at $a - \omega$ is equal to the sine of an arc terminated at A $+\ \omega$, or at A$' - \omega$, and has the same sign, and differs only in sign from the sine of an arc terminated at A $+\ \omega$ or A$' - \omega$; and that the cosine of an arc terminated at $a + \omega$ differs only in sign from the sine of an arc terminated at A $+\ \omega$, or at A$' - \omega$, and is equal to that of an arc terminated at A $-\ \omega$, or at A$' + \omega$, with the

same signs. These results are evident from the construction marked by dotted lines in Tab. III.

Further, the sine of an arc terminated at $a' \pm \omega$ is equal to the cosine of an arc terminated at $A \pm \omega$, but has a different sign, and is equal to the cosine of an arc terminated at $A' \pm \omega$, and has the same sign. Also the cosine of an arc terminated at $a' + \omega$ is the same with the same sign as the sine of an arc terminated at $A + \omega$, or $A' - \omega$, and differs only in sign from the sine of an arc terminated at $A - \omega$, or $A' + \omega$. Also the cosine of an arc terminated at $a' - \omega$ is equal to the sine of an arc terminated at $A + \omega$, or at $A' - \omega$, but has a different sign, and is the same with the same sign as the sine of an arc terminated at $A - \omega$, or at $A' + \omega$.

These several results may be expressed as follows:

$$\sin.[(2n+\tfrac{1}{2})\pi \pm \omega] = \cos.(2n\pi \pm \omega) = -\cos.[(2n+1)\pi \pm \omega],$$

$$\cos.[(2n+\tfrac{1}{2})\pi - \omega] = \sin.(2n\pi + \omega) = -\sin.\ (2n\pi - \omega)$$
$$= -\sin.[(2n+1)\pi + \omega] = \sin.[(2n+1)\pi - \omega],$$

$$\cos.[(2n+\tfrac{1}{2})\pi + \omega] = -\sin.(2n\pi + \omega) = \sin.(2n\pi - \omega)$$
$$= \sin.[(2n+1)\pi + \omega] = -\sin.[(2n+1)\pi - \omega],$$

$$\sin.[(2n+\tfrac{3}{2})\pi \pm \omega = -\cos.(2n\pi \pm \omega) = \cos.(2n+1)\pi \pm \omega],$$

$$\cos.[(2n+\tfrac{3}{2})\pi + \omega] = \sin.(2n\pi + \omega) = -\sin.(2n\pi - \omega)$$
$$= \sin.[(2n+1)\pi - \omega] = -\sin.[(2n+1)\pi + \omega],$$

$$\cos.[(2n+\tfrac{3}{2})\pi - \omega] = -\sin.(2n\pi + \omega) = \sin.(2n\pi - \omega)$$
$$= -\sin.[(2n+1)\pi - \omega] = \sin.[(2n+1)\pi + \omega]\ *.$$

(41.) The relations between the sines and cosines of angles of these forms necessarily (by Tab. II.) determine the re-

---

* The cases of these formulæ most frequently used are

$$\sin.\left(\frac{\pi}{2} \pm \omega\right) = \cos.\omega.$$

$$\cos.\left(\frac{\pi}{2} \pm \omega\right) = \mp \sin.\omega.$$

lations between all the other trigonometrical terms. It is therefore unnecessary to enter upon any similar investigations concerning the tangents, cotangents, &c. as it will be easy to derive their relations when required from those already determined.

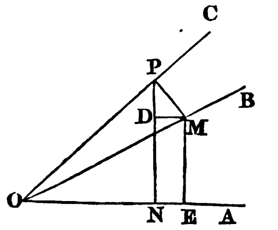
SECTION III.

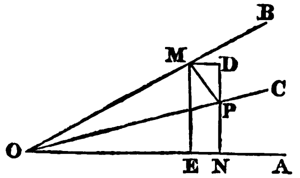
On the relations between angles and their sums and differences.

(42.) From the definition of the sine of an angle or arc, it appears that twice the sine of any arc is equal to the chord of double that arc. For if the sine PM in the fig. Tab. III. be produced to meet the circle, it will cut off an arc $PP' = 2PA$. This principle furnishes an easy solution of the following problem.

PROP. VII.

(43.) *Given the sines and cosines of two arcs, to find the sine and cosine of their sum and difference.*





Let AOB be the greater angle ω, and BOC the lesser ω'. Then AOC will be the sum $\omega + \omega'$, or the difference $\omega - \omega'$, according as OC and OA lie at different sides of OB, or at the same side. Take any point P on OC and draw PM and PN perpendicular to OB and OA, and from M draw ME perpendicular to OA. Through M draw MD parallel

to OA to meet NP, produced, if necessary, at D. By the definitions in (21) we have

$$\sin.\omega = \frac{ME}{MO}, \quad \sin.\omega' = \frac{PM}{PO} \quad \sin.(\omega \pm \omega') = \frac{PN}{PO}.$$

$$\cos.\omega = \frac{EO}{MO}, \quad \cos.\omega' = \frac{MO}{PO} \quad \cos.(\omega \pm \omega') = \frac{NO}{PO}.$$

But PN = DN $\pm$ DP. Hence

$$\sin.(\omega \pm \omega') = \frac{DN}{PO} \pm \frac{DP}{PO} = \frac{ME}{PO} \pm \frac{DP}{PO}$$

$$= \frac{ME}{MO} \cdot \frac{MO}{PO} \pm \frac{DP}{PM} \cdot \frac{PM}{PO}.$$

By the similar triangles $\frac{DP}{PM} = \frac{EO}{MO}$. Hence

$$\sin.(\omega \pm \omega') = \sin.\omega\cos.\omega' \pm \sin.\omega'\cos.\omega \quad \cdots\cdots \quad [1].$$

In like manner NO = OE $\mp$ NE = OE $\mp$ DM. Hence

$$\cos.(\omega \pm \omega') = \frac{OE}{PO} \mp \frac{DM}{PO}$$

$$= \frac{OE}{OM} \cdot \frac{OM}{PO} \mp \frac{DM}{PM} \cdot \frac{PM}{PO}.$$

By similar triangles $\frac{DM}{PM} = \frac{ME}{MO}$. Hence

$$\cos.(\omega \pm \omega') = \cos.\omega\cos.\omega' \mp \sin.\omega\sin.\omega'. \quad \cdots\cdots \quad [2].$$

In this investigation the upper sign refers to the first figure, and the lower to the second.

(44.) From the two formulæ [1] may be deduced a group of four others by merely adding, subtracting, multiplying, and dividing them.

1°. By adding them, we obtain

$$\sin.(\omega + \omega') + \sin.(\omega - \omega') = 2\sin.\omega\cos.\omega' \quad \cdots\cdots \quad [3].$$

2°. By subtracting,

$$\sin.(\omega + \omega') - \sin.(\omega - \omega') = 2\sin.\omega'\cos.\omega \quad \cdots\cdots \quad [4].$$

3°. By multiplying,

$$\sin.(\omega + \omega')\sin.(\omega - \omega') = \sin.^2\omega \cos.^2\omega' - \sin.^2\omega' \cos.^2\omega.$$

Eliminating $\cos.^2\omega'$ and $\cos.^2\omega$ by the equations

$$\sin.^2\omega + \cos.^2\omega = 1,$$
$$\sin.^2\omega' + \cos.^2\omega' = 1,$$

the result is

$$\sin.(\omega + \omega')\sin.(\omega - \omega') = \sin.^2\omega - \sin.^2\omega' \quad . \quad . \quad . \quad [5].$$

Whence may be inferred, that "*the product of the sines of the sum and difference of two angles is equal to the product of the sum and difference of their sines.*"

4°. By dividing,

$$\frac{\sin.(\omega + \omega')}{\sin.(\omega - \omega')} = \frac{\sin.\omega \cos.\omega' + \sin.\omega' \cos.\omega}{\sin.\omega \cos.\omega' - \sin.\omega' \cos.\omega}.$$

Eliminating the sines and cosines of ω and ω' by the equations

$$\frac{\sin.\omega}{\cos.\omega} = \tan.\omega, \qquad \frac{\sin.\omega'}{\cos.\omega'} = \tan.\omega',$$

the result is

$$\frac{\sin.(\omega + \omega')}{\sin.(\omega - \omega')} = \frac{\tan.\omega + \tan.\omega'}{\tan.\omega - \tan.\omega'} \quad . \quad . \quad . \quad . \quad . \quad . \quad [6].$$

(45.) By eliminating the sines from [1] by the values of the tangents 2, Tab. II., we obtain

$$\frac{\sin.(\omega \pm \omega')}{\cos.\omega \cos.\omega'} = \tan.\omega \pm \tan.\omega' \quad . \quad . \quad . \quad . \quad . \quad [7].$$

(46.) In a similar way another group of four formulæ may be deduced from [2] by addition, subtraction, multiplication, and division.

1°. By addition,

$$\cos.)\omega + \omega') + \cos.(\omega - \omega') = 2\cos.\omega \cos.\omega' \quad . \quad . \quad . \quad [8].$$

2°. By subtraction,

$$\cos.(\omega + \omega') - \cos.(\omega - \omega') = -2\sin.\omega \sin.\omega' \quad . \quad . \quad . \quad [9].$$

3°. By multiplication,

$$\cos.(\omega + \omega')\cos.(\omega - \omega') = \cos.^2\omega \cos.^2\omega' - \sin.^2\omega \sin.^2\omega'.$$

Eliminating $\sin.^2\omega$, $\sin.^2\omega'$, by the equations

$$\cos.^2\omega + \sin.^2\omega = 1,$$
$$\cos.^2\omega' + \sin.^2\omega' = 1,$$

we obtain

$$\cos.(\omega + \omega')\cos.(\omega - \omega') = \cos.^2\omega + \cos.^2\omega' - 1 \quad \cdot \cdot \cdot \quad [10].$$

or by eliminating $\cos.^2\omega$, $\cos.^2\omega'$, we have

$$\cos.(\omega + \omega')\cos.(\omega - \omega') = 1 - \sin.^2\omega - \sin.^2\omega'.$$

4°. By division,

$$\frac{\cos.(\omega+\omega')}{\cos.(\omega-\omega')} = \frac{\cos.\omega\cos.\omega' - \sin.\omega\sin.\omega'}{\cos.\omega\cos.\omega' + \sin.\omega\sin.\omega'}.$$

Eliminating the sines and cosines by

$$\frac{\cos.\omega}{\sin.\omega} = \cot.\omega, \qquad \frac{\cos.\omega'}{\sin.\omega'} = \cot.\omega',$$

we obtain

$$\frac{\cos.(\omega+\omega')}{\cos.(\omega-\omega')} = \frac{\tan.\omega' - \tan.\omega}{\tan.\omega' + \tan.\omega} \quad \cdot \cdot \cdot \cdot \cdot \quad [11].$$

(47.) By eliminating the cosines from [2] by the values for the cotangents 3, Tab. II., we obtain

$$\frac{\cos.(\omega \pm \omega')}{\sin.\omega.\sin.\omega'} = \cot.\omega\cot.\omega' \mp 1 \quad \cdot \cdot \cdot \cdot \quad [12].$$

PROP. VIII.

(48.) *Given the tangents of two angles, to determine the tangents of their sum and difference.*

By dividing [1] by [2], we obtain,

$$\frac{\sin.(\omega \pm \omega')}{\cos.(\omega \pm \omega')} = \frac{\sin.\omega\cos.\omega' \pm \sin.\omega'\cos.\omega}{\cos.\omega\cos.\omega' \mp \sin.\omega\sin.\omega'}.$$

Eliminating the sines and cosines by the values for the tangents in 2, Tab. II., we find

$$\tan.(\omega \pm \omega') = \frac{\tan.\omega \pm \tan.\omega'}{1 \mp \tan.\omega\tan.\omega'} \quad \cdot \cdot \cdot \cdot \cdot \quad [13]$$

PROP. IX.

(49.) *Given the cotangents of two angles, to find the cotangents of their sum and difference.*

Let [2] be divided by [1], and the sines and cosines eliminated by the values of the cotangents in 3, Tab. II., and the result is

$$\cot.(\omega \pm \omega') = \frac{\cot.\omega \cot.\omega' \mp 1}{\cot.\omega' \pm \cot.\omega} \quad \ldots\ldots [14].$$

PROP. X.

(50.) *Given the secants and tangents of two angles, to find the secants of their sum and difference.*

Taking the reciprocals of [2], and eliminating the sines and cosines by 2 and 5, Tab. II., the result is

$$\sec.(\omega \pm \omega') = \frac{\sec.\omega \sec.\omega'}{1 \mp \tan.\omega \tan.\omega'} \quad \ldots\ldots [15].$$

PROP. XI.

(51.) *Given the cosecants and cotangents of two angles, to find the cosecant of their sum and difference.*

Taking the reciprocals of [1], and eliminating the sines and cosines by 3 and 6 of Tab. II., the result is

$$\text{cosec.}(\omega \pm \omega') = \frac{\text{cosec.}\omega\, \text{cosec.}\omega'}{\cot.\omega' \pm \cot.\omega} \quad \ldots\ldots [16].$$

(52.) Any trigonometrical formula expressing a relation between the sum and difference of two angles, and the angles themselves may, by a very simple transformation, be changed into one expressing a relation between two angles and their half sum and half difference. This change is effected by considering that the sum of $\omega + \omega'$ and $\omega - \omega'$ is 2ω, and

that therefore half their sum is ω. Also the difference of $\omega + \omega'$ and $\omega - \omega'$, is $2\omega'$, and therefore half their difference is ω'.

If then in any formula expressing a relation between $\omega + \omega'$, $\omega - \omega'$, ω and ω'; $\omega + \omega'$ and $\omega - \omega'$ be considered as independent angles, ω being considered as half their sum, and ω' as half their difference, the formula will still be true. Therefore, in every such formula, we are at liberty to change $\omega + \omega'$ into ω, and $\omega - \omega'$ into ω', provided we also change ω into $\frac{1}{2}(\omega + \omega')$ and ω' into $\frac{1}{2}(\omega - \omega')$. Thus the result will be a new formula expressing a relation between ω, ω', $\frac{1}{2}(\omega + \omega')$ and $\frac{1}{2}(\omega - \omega')$.

These observations point out a method of deriving a group of *sixteen* new formulæ from those already established in this section. We shall not, however, enter into the details of these deductions, as they are not all of very general use, and the student, by those which we shall explain, and which are useful, will easily perceive the method of deriving the others.

In the formulæ [3], [4], [8], [9], by changing $\omega + \omega'$ into ω, $\omega - \omega'$ into ω', ω into $\frac{1}{2}(\omega + \omega')$ and ω' into $\frac{1}{2}(\omega - \omega')$, we obtain the following group of four formulæ:

$$\sin.\omega + \sin.\omega' = 2\sin.\tfrac{1}{2}(\omega + \omega')\cos.\tfrac{1}{2}(\omega - \omega') \quad \cdots\cdot \quad [17].$$

$$\sin.\omega - \sin.\omega' = 2\sin.\tfrac{1}{2}(\omega - \omega')\cos.\tfrac{1}{2}(\omega + \omega') \quad \cdots\cdot \quad [18].$$

$$\cos.\omega + \cos.\omega' = 2\cos.\tfrac{1}{2}(\omega + \omega')\cos.\tfrac{1}{2}(\omega - \omega') \quad \cdots\cdot \quad [19].$$

$$\cos.\omega - \cos.\omega' = -2\sin.\tfrac{1}{2}(\omega + \omega')\sin.\tfrac{1}{2}(\omega - \omega') \quad \cdots \quad [20].$$

(53.) From these four a group of six others may be immediately deduced by division. Let the first be divided successively by the second, third, and fourth; the second by the third and fourth; and the third by the fourth. After this division, the sines and cosines of $\frac{1}{2}(\omega + \omega')$ and $\frac{1}{2}(\omega - \omega')$ being eliminated by the formulæ 2 and 3, Tab. II. we obtain the following formulæ:

$$\frac{\sin.\omega + \sin.\omega'}{\sin.\omega - \sin.\omega'} = \frac{\tan.\frac{1}{2}(\omega + \omega')}{\tan.\frac{1}{2}(\omega - \omega')} \quad . \quad . \quad . \quad . \quad . \quad . \quad [21].$$

$$\frac{\sin.\omega + \sin.\omega'}{\cos.\omega + \cos.\omega'} = \tan.\tfrac{1}{2}(\omega + \omega') \quad . \quad . \quad . \quad . \quad . \quad . \quad [22].$$

$$\frac{\sin.\omega + \sin.\omega'}{\cos.\omega - \cos.\omega'} = -\cot.\tfrac{1}{2}(\omega - \omega') \quad . \quad . \quad . \quad . \quad . \quad [23].$$

$$\frac{\sin.\omega - \sin.\omega'}{\cos.\omega + \cos.\omega'} = \tan.\tfrac{1}{2}(\omega - \omega') \quad . \quad . \quad . \quad . \quad . \quad . \quad [24].$$

$$\frac{\sin.\omega - \sin.\omega'}{\cos.\omega - \cos.\omega'} = -\cot.\tfrac{1}{2}(\omega + \omega') \quad . \quad . \quad . \quad . \quad . \quad [25].$$

$$\frac{\cos.\omega + \cos.\omega'}{\cos.\omega - \cos.\omega'} = -\cot.\tfrac{1}{2}(\omega + \omega')\cot.\tfrac{1}{2}(\omega - \omega') \quad [26].$$

It is unnecessary to extend our enumeration of the formulæ relative to the sums and differences of angles farther, as we have deduced all that are of any use in the applications of the science, and from those which have been established, all others may without difficulty be inferred.

(54.) The results of this and the following sections with some additions will be found in Table IV. at the end of this volume. Annexed to each formula is an indication of the operation by which it is obtained. The most important formulæ are marked *.

SECTION IV.

On the relations between the sines, cosines, &c. of angles, and those of their doubles and halves. Trigonometrical terms connected with particular angles.

PROP. XII.

(55.) *To determine the sine and cosine of double a given angle.*

If, in [1], $\omega = \omega'$, we obtain

$$\sin.2\omega = 2\sin.\omega \cos.\omega \quad \cdots\cdots \quad [27].$$

If in [2] $\omega = \omega'$, we find

$$\cos.2\omega = \cos.^2\omega - \sin.^2\omega \quad \cdots\cdots \quad [28],$$

which, by 1, Tab. II. may be expressed in either of the following ways:

$$\cos.2\omega = 2\cos.^2\omega - 1 \quad \cdots\cdots \quad [29],$$

$$\cos.2\omega = 1 - 2\sin.^2\omega \quad \cdots\cdots \quad [30].$$

The last two may be also deduced from [10], by supposing $\omega = \omega'$.

PROP. XIII.

(56.) *To determine the tangent and cotangent of double a given angle.*

By supposing $\omega = \omega'$ in [13], we obtain

$$\tan.2\omega = \frac{2\tan.\omega}{1-\tan.^2\omega} \quad \cdots\cdots \quad [31].$$

$$= \frac{2\cot.\omega}{\cot.^2\omega - 1} \quad \cdots\cdots \quad [32].$$

By dividing both the numerator and denominator by $\cot.\omega$, we find

$$\tan.2\omega = \frac{2}{\cot.\omega - \tan.\omega} \quad \cdots\cdots \quad [33].$$

By making $\omega = \omega'$ in [14], we obtain

$$\cot.2\omega = \frac{\cot.^2\omega - 1}{2\cot.\omega} \quad \cdots\cdots \quad [34],$$

$$= \tfrac{1}{2}(\cot.\omega - \tan.\omega) \quad \cdots \quad [35].$$

PROP. XIV.

(57.) *To determine the secant and cosecant of double a given angle.*

By making $\omega = \omega'$ in [15], we find

$$\sec.2\omega = \frac{\sec.^2\omega}{1-\tan.^2\omega} \quad . \quad . \quad . \quad . \quad . \quad . \quad [36],$$

$$= \frac{1+\tan.^2\omega}{1-\tan.^2\omega} \quad . \quad . \quad . \quad . \quad . \quad . \quad [37],$$

$$= \frac{\text{cosec.}^2\omega}{\cot.^2\omega-1} \quad . \quad . \quad . \quad . \quad . \quad . \quad [38],$$

$$= \frac{\cot.^2\omega+1}{\cot.^2\omega-1} \quad . \quad . \quad . \quad . \quad . \quad . \quad [39].$$

In like manner, by [16],

$$\text{cosec.}2\omega = \frac{\text{cosec.}^2\omega}{2\cot.\omega},$$

$$= \frac{\cot.^2\omega+1}{2\cot.\omega},$$

$$= \tfrac{1}{2}(\tan.\omega + \cot.\omega) \quad . \quad . \quad . \quad [40].$$

PROP. XV.

(58.) *To determine the sine and cosine of half of a given angle.*

In [30] let 2ω be changed into ω, and ω into $\frac{1}{2}\omega$; since the relative magnitude of the angles remains unchanged, the formula is still true, $\therefore$

$$\cos.\omega = 1 - 2\sin.^2\tfrac{1}{2}\omega,$$

$$\therefore \sin.^2\tfrac{1}{2}\omega = \tfrac{1}{2}(1 - \cos.\omega) \quad . \quad . \quad [41].$$

A similar transformation in [29] gives

$$\cos.\omega = 2\cos.^2\tfrac{1}{2}\omega - 1,$$

$$\therefore \cos.^2\tfrac{1}{2}\omega = \tfrac{1}{2}(1 + \cos.\omega) \quad . \quad . \quad . \quad [42].$$

(59.) *Cor.* The sine of an angle may be expressed in terms of its half by making a similar change in [27], which gives

$$\sin.\omega = 2\sin.\tfrac{1}{2}\omega\cos.\tfrac{1}{2}\omega.$$

PROP. XVI.

(60.) *To determine the tangent and cotangent of half of a given angle.*

By supposing $\omega' = 0$ in [22], we obtain

$$\tan.\tfrac{1}{2}\omega = \frac{\sin.\omega}{1+\cos.\omega} \quad . \quad . \quad . \quad . \quad . \quad . \quad [43].$$

The same might be obtained by dividing [41] by [42], which gives

$$\tan.^2\tfrac{1}{2}\omega = \frac{1-\cos.\omega}{1+\cos.\omega} \quad . \quad . \quad . \quad . \quad . \quad . \quad [44].$$

The numerator and denominator of this being multiplied by $1+\cos.\omega$, gives [43], and being multiplied by $1-\cos.\omega$, gives

$$\tan.\tfrac{1}{2}\omega = \frac{1-\cos.\omega}{\sin.\omega} \quad . \quad . \quad . \quad . \quad . \quad . \quad [45],$$

observing the condition, 1 Tab. II.

By eliminating the sines and cosines from the last three formulæ, we find

$$\tan.\tfrac{1}{2}\omega = \frac{\tan.\omega}{\sec.\omega+1} \quad . \quad . \quad . \quad . \quad . \quad . \quad [46].$$

$$\tan.\tfrac{1}{2}\omega = \frac{\sec.\omega-1}{\tan.\omega}.$$

$$\tan.^2\tfrac{1}{2}\omega = \frac{\sec.\omega-1}{\sec.\omega+1} \quad . \quad . \quad . \quad . \quad . \quad . \quad [47].$$

$$\tan.\tfrac{1}{2}\omega = \text{cosec.}\omega - \cot.\omega \quad . \quad . \quad . \quad [48].$$

By taking the reciprocals of the last six formulæ, we find

$$\cot.\tfrac{1}{2}\omega = \frac{1+\cos.\omega}{\sin.\omega} \quad . \quad . \quad . \quad . \quad . \quad . \quad [49].$$

$$\cot.^2\tfrac{1}{2}\omega = \frac{1+\cos.\omega}{1-\cos.\omega} \quad . \quad . \quad . \quad . \quad . \quad . \quad [50].$$

$$\cot.\tfrac{1}{2}\omega = \frac{\sin.\omega}{1-\cos.\omega} \quad . \quad . \quad . \quad . \quad . \quad . \quad [51].$$

$$\cot.\tfrac{1}{2}\omega = \frac{\sec.\omega+1}{\tan.\omega} \quad . \quad . \quad . \quad . \quad . \quad . \quad [52].$$

$$\cot.^2\tfrac{1}{2}\omega = \frac{\sec.\omega+1}{\sec.\omega-1} \quad . \quad . \quad . \quad . \quad . \quad . \quad [53].$$

$$\cot.\tfrac{1}{2}\omega = \text{cosec.}\omega + \cot.\omega \quad . \quad . \quad . \quad [54].$$

PROP. XVII.

(61.) *To determine the secant and cosecant of half of given angle.*

Let the reciprocals of [42] and [41] be taken, and tl sines and cosines eliminated by the formulæ of Tab. II. ar the results are

$$\sec.^2\tfrac{1}{2}\omega = \frac{2\sec.\omega}{\sec.\omega + 1} \quad . \quad . \quad . \quad . \quad . \quad . \quad [55].$$

$$\text{cosec.}^2\tfrac{1}{2}\omega = \frac{2\sec.\omega}{\sec.\omega - 1} \quad . \quad . \quad . \quad . \quad . \quad . \quad [56].$$

PROP. XVIII.

(62.) *To determine the sine, cosine, &c. of* $\frac{\pi}{4}$ or 45°.

If, in the equation,

$$\sin.^2\omega + \cos.^2\omega = 1,$$

$$\text{or } \sin.^2\omega + \sin.^2\left(\frac{\pi}{2} - \omega\right) = 1,$$

$\frac{\pi}{4}$ be substituted for ω, it becomes

$$2\sin.^2\frac{\pi}{4} = 1,$$

$$\therefore \sin.\frac{\pi}{4} = \frac{1}{\sqrt{2}} \therefore \cos.\frac{\pi}{4} = \frac{1}{\sqrt{2}}.$$

Hence, by Tab. II.

$$\tan.\frac{\pi}{4} = 1, \quad \cot.\frac{\pi}{4} = 1,$$

$$\sec.\frac{\pi}{4} = \sqrt{2}, \text{ cosec.}\frac{\pi}{4} = \sqrt{2}.$$

PROP. XIX.

(63.) *To determine the sine, cosine, &c. of* $\frac{\pi}{3}$ or 60°.

If, in the formula,

$$\sin.2\omega = 2\sin.\omega\cos.\omega,$$

30° be substituted for ω, it becomes

$$\sin.60^\circ = 2\sin.30^\circ\cos.30^\circ;$$

but 30° is the complement of 60°,

$$\therefore \sin.30^\circ = \cos.60^\circ, \quad \cos.30^\circ = \sin.60^\circ.$$

Hence

$$\sin.60^\circ = 2\cos.60^\circ\sin.60^\circ,$$

$$\therefore \cos.60^\circ = \frac{1}{2}.$$

Hence by 1, Tab. II.

$$\sin.60^\circ = \frac{\sqrt{3}}{2};$$

and by 4, 5, 6, Tab. I.

$$\tan.60^\circ = \sqrt{3}, \quad \cot.60^\circ = \frac{1}{\sqrt{3}},$$

$$\sec.60^\circ = 2, \quad \operatorname{cosec}.60^\circ = \frac{2}{\sqrt{3}}.$$

PROP. XX.

(64.) *To determine the sine, cosine, &c. of* $\frac{\pi}{6}$ or 30°.

This being the complement of 60°, we have at once

$$\sin.30^\circ = \frac{1}{2}, \quad \cos.30^\circ = \frac{\sqrt{3}}{2},$$

$$\tan.30^\circ = \frac{1}{\sqrt{3}}, \quad \cot.30^\circ = \sqrt{3},$$

$$\sec.30^\circ = \frac{2}{\sqrt{3}}, \quad \operatorname{cosec}.30^\circ = 2.$$

(65.) The peculiar values of the sines, cosines, &c. of these angles, 45°, 60°, 30°, produce remarkable modifications in the formulæ in which they enter. Instances of these may be seen in Tab. IV. from 50 to the end.

SECTION V.

On the solution of plane triangles.

(66.) In a plane triangle there are six parts, the three sides and the three angles. In general, if the magnitudes of any three of these six quantities be given, the magnitudes of the other three may be computed by the aid of the formulæ and tables of trigonometry. We say, *in general*, because there are particular cases where the question becomes indeterminate, or the result equivocal; as in the case where the data are the three angles, in which, though the sides cannot be computed, yet their proportion may, and the triangle may be determined *in species*.

In the resolution of triangles there are two distinct processes to be attended to. The first is to establish general formulæ exhibiting in every case the relation of the parts of the triangle which are sought to those which are given, so that the former may be computed from the latter. This shall form the subject of the present section.

The second is the calculation of certain numerical tables, in which the values of the sines, cosines, &c. of angles, or the logarithms of these numbers, are registered; and by reference to which, the computations indicated by the general formulæ before mentioned may be effected. This subject, not being absolutely necessary in the more elementary parts of science, we shall reserve for a subsequent part of this work.

To guide us, in some degree, however, in the determination of formulæ exhibiting the relations of the parts of a triangle, it is necessary that we should attend to the nature of the tables, by the aid of which these formulæ are to be computed when particular numbers are substituted for their general symbols. These tables are in general logarithmic *, and by them, whenever an angle is known, the logarithms of its sine, cosine, tangent, &c. can be determined, and, *vice versâ*, when any of the latter are *known*, the angle can be found. Formulæ, therefore, in order to be suited to such tables, should be such as are adapted for logarithmic calculation, and therefore their different parts should be united as much as possible by multiplication, division, involution, and evolution; and as little as possible by addition and subtraction.

Further, as it would be impossible in any tables to give the values of the sine, cosine, &c. of angles of all magnitudes, it must frequently happen that the angle, sine, or cosine, &c. which we seek, lies between two successive tabulated angles, sines, or cosines, &c. We can in this case only compute the true value approximately, and the degree of the approximation frequently depends on the formula which we use. A formula which is proper when the angle is great, is often ill suited if the angle to be computed be small. Hence it is necessary in some cases to establish several different formulæ for the solution of the same problem, some being fitted for calculation in the cases where others fail, or give results deviating considerably from the truth.

---

* It is necessary that the student should be acquainted with the properties of logarithms, explained in the note at the conclusion of this section.

We shall first investigate the formulæ for the solution of right-angled, and next of oblique-angled triangles.

I.

The solution of right-angled triangles.

(67.) Of the six parts of a triangle, it is only necessary to consider four in the case of a right-angled triangle; since the right angle is always given, and one of the acute angles is the complement of the other.

Let a and b be the sides, c the hypothenuse, and A the angle opposite the side a.

By (21.) we have

$$\frac{a}{c} = \sin.A, \quad \frac{b}{c} = \cos.A,$$

$$\because \frac{a}{b} = \tan.A, \quad \frac{b}{a} = \cot.A;$$

and by plane geometry,

$$a^2 + b^2 = c^2.$$

These equations are sufficient for the solution of all cases of right-angled triangles. All questions as to their solution must come under some of the following four:

(68.) 1°. *Given the two sides, to find the hypothenuse and either angle.*

$$c = \sqrt{a^2 + b^2}, \quad \because lc = \tfrac{1}{2}l(a^2 + b^2) \text{ *},$$

$$\tan.A = \frac{a}{b}, \quad \because l.\tan.A = la - lb.$$

(69.) 2°. *Given the hypothenuse and one side, to find the other side and either angle.*

$$a = \sqrt{c^2 - b^2}, \because la = \tfrac{1}{2}l(c + b) + \tfrac{1}{2}l(c - b),$$

* The radius should be introduced previous to calculation when it is not = 1, or when its log. is not = 0.

$$\sin.\text{B} = \frac{b}{c}, \qquad \therefore l.\sin.\text{B} = lb - lc,$$

$$\cos.\text{A} = \frac{b}{c}, \qquad \therefore l\cos.\text{A} = lb - lc.$$

(70.) 3°. *Given the hypothenuse and one angle, to find the two sides.*

$$b = c \sin.\text{B}, \qquad a = c \cos.\text{B},$$

$$\therefore lb = lc + l \sin.\text{B}, \qquad la = lc + l \cos.\text{B}.$$

(71.) 4°. *Given either side and an angle, to find the hypothenuse and the other side.*

$$c = \frac{b}{\sin.\text{B}} = \frac{b}{\cos.\text{A}}, \quad a = \frac{b}{\tan.\text{B}} = \frac{b}{\cot.\text{A}},$$

$$\therefore lc = lb - l \sin.\text{B} = lb - l \cos.\text{A},$$

$$la = lb - l \tan.\text{B} = lb - l \cot.\text{A}.$$

II.

The solution of oblique angled triangles.

(72.) Let a, b, c, be the three sides, and A, B, C, the angles opposed to them respectively. Let p be the perpendicular from any angle A on the opposite side a. We have obviously

$$\sin.\text{B} = \frac{p}{c}, \quad \sin.\text{C} = \frac{p}{b}.$$

Eliminating p, we obtain

$$\frac{\sin.\text{B}}{\sin.\text{C}} = \frac{b}{c},$$

and, in like manner,

$$\frac{\sin.\text{C}}{\sin.\text{A}} = \frac{c}{a}, \quad \frac{\sin.\text{A}}{\sin.\text{B}} = \frac{a}{b}.$$

Hence "the sides of a plane triangle are as the sines of the opposite angles."

(73.) From the formula

$$\frac{\sin.A}{\sin.B}=\frac{a}{b},$$

we deduce

$$\frac{\sin.A+\sin.B}{\sin.A-\sin.B}=\frac{a+b}{a-b},$$

which by [21] becomes

$$\frac{a+b}{a-b}=\frac{\tan.\frac{1}{2}(A+B)}{\tan.\frac{1}{2}(A-B)}.$$

Hence "the sum of two sides of a plane triangle is to their difference as the tangent of half the sum of the opposite angles is to the tangent of half their difference." This formula is independent of the absolute magnitudes of the sides, and will be applicable if their ratio only be expressed by $a : b$.

(74.) By [17] and [18] we have

$$\sin.A + \sin.B = 2\sin.\tfrac{1}{2}(A + B)\cos.\tfrac{1}{2}(A - B),$$
$$\sin.A - \sin.B = 2\sin.\tfrac{1}{2}(A - B)\cos.\tfrac{1}{2}(A + B).$$

But since $A + B + C = \pi$, $\because$

$$\sin.\tfrac{1}{2}(A + B) = \cos.\tfrac{1}{2}C,$$
$$\cos.\tfrac{1}{2}(A + B) = \sin.\tfrac{1}{2}C.$$

Also (72.),

$$\sin.A = \frac{a}{c}\sin.C,$$

$$\sin.B = \frac{b}{c}\sin.C.$$

Hence we obtain

$$\frac{a+b}{c}\sin.C = 2\cos.\tfrac{1}{2}C\cos.\tfrac{1}{2}(A - B),$$

$$\frac{a-b}{c}\sin.C = 2\sin.\tfrac{1}{2}C\sin.\tfrac{1}{2}(A - B).$$

But since [27],

$$\sin.C = 2\sin.\tfrac{1}{2}C\cos.\tfrac{1}{2}C;$$

we find, by substituting and dividing by the common multiplier,

$$\cos.\tfrac{1}{2}(\text{A} - \text{B}) = \frac{a+b}{c}\sin.\tfrac{1}{2}\text{C},$$

$$\sin.\tfrac{1}{2}(\text{A} - \text{B}) = \frac{a-b}{c}\cos.\tfrac{1}{2}\text{C}.$$

It is obvious that these formulæ are also independent of the absolute magnitudes of the sides, and merely depend on their proportion.

By dividing these formulæ each by the other, we obtain

$$\tan.\tfrac{1}{2}(\text{A} - \text{B}) = \frac{a-b}{a+b}\cot.\tfrac{1}{2}\text{C},$$

$$\cot.\tfrac{1}{2}(\text{A} - \text{B}) = \frac{a+b}{a-b}\tan.\tfrac{1}{2}\text{C}.$$

(75.) Let s be the segment of the side b between the perpendicular p and the angle C. By the definitions (21) we have

$$s = a\cos.\text{C}.$$

But also

$$c^2 = p^2 + (b - s)^2$$

$$\therefore c^2 = p^2 + b^2 + s^2 - 2bs.$$

But $a^2 = p^2 + s^2 \therefore$

$$c^2 = a^2 + b^2 - 2ab\cos.\text{C}.$$

Hence " the square of any side of a plane triangle is equal to the sum of the squares of the two remaining sides diminished by twice the rectangle under them multiplied by the cosine of the included angle *."

---

* Eucl. lib. ii. prop. 12, 13.

(76.) By adding the equations

$$c^2 = a^2 + b^2 - 2ab\cos.\mathrm{c},$$
$$0 = 2ab - 2ab,$$

we obtain

$$c^2 = (a+b)^2 - 2ab(1+\cos.\mathrm{c}),$$
$$\text{or } c^2 = (a-b)^2 + 2ab(1-\cos.\mathrm{c}).$$

Hence by [42] and [41], Sect. IV.

$$4ab\cos.^2\tfrac{1}{2}\mathrm{c} = (a+b)^2 - c^2,$$
$$4ab\sin.^2\tfrac{1}{2}\mathrm{c} = -(a-b)^2 + c^2.$$

By multiplying and dividing these, observing the equation

$$\sin.\mathrm{c} = 2\sin.\tfrac{1}{2}\mathrm{c}\cos.\tfrac{1}{2}\mathrm{c},$$

we obtain

$$4a^2b^2\sin.^2\mathrm{c} = [(a+b)^2 - c^2] \times [-(a-b)^2 + c^2],$$

$$\tan.^2\tfrac{1}{2}\mathrm{c} = \frac{-(a-b)^2+c^2}{(a+b)^2-c^2}.$$

All these results may be easily adapted for logarithmic calculation. Let

$$2s = a + b + c,$$
$$\because 2(s-a) = b + c - a,$$
$$2(s-b) = a + c - b,$$
$$2(s-c) = b + a - c.$$

Also

$$(a+b)^2 - c^2 = (a+b+c)(a+b-c),$$
$$-(a-b)^2 + c^2 = (a+c-b)(c+b-a).$$

Hence the four equations already obtained, when divided by $4ab$, give

$$\cos.^2\tfrac{1}{2}\mathrm{c} = \frac{s(s-c)}{ab},$$

$$\sin.^2\tfrac{1}{2}C = \frac{(s-a)(s-b)}{ab}$$

$$\sin.^2 C = \frac{4s.(s-a)(s-b)(s-c)}{a^2b^2},$$

$$\tan.^2\tfrac{1}{2}C = \frac{(s-a)(s-b)}{s(s-c)},$$

which are all suited to logarithmic computation.

(77.) If L be the area of the triangle, it is evident that

$$2L = pa.$$

But $p = b\sin.C$, $\because$

$$2L = ab\sin.C.$$

Hence " twice the area of a triangle is equal to the rectangle under two sides multiplied into the sine of the included angle."

By the third of the last four formulæ, $\because$ we obtain

$$L = \sqrt{s.(s-a)(s-b)(s-c)}.$$

Hence " the area of a triangle is equal to the square root of the continued product of its semiperimeter, and the differences between the semiperimeter and each of the three sides."

By this formula the area of a triangle may be computed by logarithms when its sides are given.

(78.) By applying the fourth formula found in (76.) to the angle B, we have

$$\tan.^2\tfrac{1}{2}B = \frac{(s-a)(s-c)}{s(s-b)}.$$

Let this be multiplied by the value of $\tan.^2\frac{1}{2}C$, and the square root of the result taken, and we find

$$\tan.\tfrac{1}{2}B \tan.\tfrac{1}{2}C = \frac{s-a}{s},$$

$$\because \tan.\tfrac{1}{2}B = \cot.\tfrac{1}{2}C \,.\, \frac{s-a}{s}.$$

Also, by dividing by the value of $\tan.^2\frac{1}{2}C$, we find

$$\frac{\tan.\frac{1}{2}B}{\tan.\frac{1}{2}C} = \frac{s-c}{s-b}.$$

(79.) Of the six quantities which compose an oblique angled triangle, it is unnecessary to consider more than five, since any one of the angles is the supplement of the sum of the other two. Of the three sides and two angles, any three being given, the other two may be computed. All the problems, therefore, which can be proposed respecting oblique angled triangles may be found by assuming every different combination of three data which can be obtained from the five parts. They may all be reduced to the following:

1° Given two sides and the angle opposite to one of them.

2°. Given two sides and the angle included by them.

3°. Given two angles and the side opposite to one of them.

4°. Given two angles and the side between them.

5°. Given the three sides.

We shall consider these problems successively.

PROP. XXI.

(80.) *Given two sides of a triangle and the angle opposite to one of them, to compute the angle opposite to the other, and the remaining side.*

Let a and b be the given sides, and A the given angle.

By (72.) we have

$$\sin.B = \frac{b}{a}\sin.A,$$

$$\therefore l\sin.B = lb + l\sin.A - la.$$

By this formula, sin.B becomes known, but, except in certain cases, the angle B will be equivocal. Since an angle and its supplement have the same sine, and neither necessarily exceeding two right angles, they may be each an angle of a triangle; therefore we can only determine that B is either

of two angles which are known and supplemental, but it is in general impossible to decide which it is.

If $b < a$, the angle B must be acute. It may therefore be found in this case.

If it happen to be known that $b < c$, though c itself be not known, the same conclusion follows.

The angle B being computed, the angle C becomes known, and thence the side c may be computed by

$$c = \frac{\sin.C}{\sin.A} \cdot a.$$

But it is frequently desirable to compute the side c immediately and independently of the angles B and C.

By (75.) we have

$$c = b\cos.A + a\cos.B,$$

$$= b\cos.A \pm a\sqrt{1 - \sin.^2 B};$$

but $$\sin.^2 B = \frac{b^2}{a^2}\sin.^2 A,$$

$$\therefore c = b\cos.A \pm \sqrt{a^2 - b^2\sin.^2 A}.$$

This, as well as the former, is equivocal and owing to the same cause. The sign $-$ is to be taken when $B > 90^\circ$, and $+$ when $B < 90^\circ$. This is evident since the sign of the cosine of an obtuse angle is negative, and that of an acute angle positive (36.).

PROP. XXII.

(81.) *Given two sides and the included angle, to compute the remaining angles and the third side.*

To compute the angles, it is only necessary that the *ratio* of the two sides be known, as is evident from Euc. lib. vi. prop. 6. Let this ratio be $a : b$. By (74.) we have

$$\therefore \tan.\tfrac{1}{2}(A - B) = \frac{a-b}{a+b}\cot.\tfrac{1}{2}C,$$

$$\text{or } \cot.\tfrac{1}{2}(A - B) = \frac{a+b}{a-b}\tan.\tfrac{1}{2}C.$$

E

$$c = a\,\frac{\sin.(A+B)}{\sin.A}.$$

The angle C is determined by

$$C = \pi - A - B.$$

PROP. XXIV.

(83.) *Given two angles and the side between them, to compute the remaining sides.*

Since the two given angles determine the third angle, this proposition is reduced to the last.

PROP. XXV.

(84.) *Given the three sides, to compute the angles.*

This problem may be solved by any of the four formulæ determined in (76.), and which are all suited to logarithms.

(85.) We conclude then, in general, that if any three of the six quantities engaged in a plane triangle be given, one of those three at least being a side, the other three may be computed.

Note on Logarithms. Art. (66.).

(86.) It appears from the results of the present section, that the solution of triangles involves the operations of multiplication, division, involution, and evolution of the arithmetical values of the trigonometrical quantities engaged in the formulæ which furnish rules for these solutions. Thus, for example, when two sides of a triangle a and b are given,

35.467328 feet, and 541.237439 feet, and the angle A opposite a is given, 18° 4′ 8″.5, to determine the sine of the angle B, the formula

$$\sin.B = \frac{b}{a}\sin.A$$

becomes

$$\sin.B = \frac{541.237439}{35.467328} \times \sin.(18^\circ\ 4'\ 8''.5).$$

The value of sin.(18° 4′ 8″.5) being obtained from the table of sines, it must be multiplied by 541.237439, and the result divided by 35.467328. This process, though not difficult in principle, would be extremely tedious in practice, besides being liable to errors of computation. These inconveniencies of calculation increase with the complexity of the formula, and would be still more sensible where (as frequently happens) roots are to be extracted, or powers obtained.

To remove these difficulties, and facilitate the processes of calculation, a method of computation has been invented, by which all results depending on multiplication and division may be obtained by the less troublesome processes of addition and subtraction; and by which multiplication and division in their turn become the means of obtaining the results of the more operose processes of involution and evolution.

(87.) The common algebraical rules for the multiplication, division, involution, and evolution of powers of the same quantity form the basis of this method. All numbers whatever are considered as powers of some number arbitrarily assumed, which is called a *base*, and the exponent of that power of the base, which is equal to any proposed number, is called the *logarithm* of that number. Thus, if we consider all numbers as powers of 10, and assume the equation

$$10^x = N,$$

any value being assigned to N, x will take a corresponding value. In this case, 10 is the base and x is the logarithm of N. If N be supposed to be 100, 1000, 10000, x will be successively 2, 3, 4, and, accordingly, the logarithms of 100, 1000, 10000, relatively to the base 10 are 2, 3, 4. In these cases, however, the logarithms are integers, which seldom occurs. It more frequently happens that there is no exact power of 10 either integral or fractional which is equal to N. We are always, however, able to compute the approximate value of x, or to obtain a value sufficiently near its true value, to give all the necessary accuracy to computation, and, indeed, to give as much accuracy to the results as they would have if we had used the exact value of x. To perceive this, it should be considered that the quantities which are generally involved in the formula to be computed, are either obtained by observation and measurement, or are derived by calculation from others which are so obtained. Now all observation or measurement is liable to a certain degree of inaccuracy or error, which must produce a greater or less effect upon the formula under computation. If then, in such cases, a value of x be obtained differing from its true value by a quantity less than the possible error arising from observation, such a value is as useful in computation as the true one; it may possibly by a compensation of error give a true result, a circumstance which could not happen were the exact value of x used.

(88.) Methods have been invented by which the value of x for any value which may be assigned to N may be computed either accurately, or as nearly as may be required. The explanation of these methods would introduce investigations unsuitable to this part of our treatise, and the student must therefore, for the present, be content to assume that the logarithms of all numbers have been computed and registered in tables, by reference to which they may be im-

mediately obtained without the labour of further calculation. We shall now explain how, by the assistance of such tables, the computations already alluded to may be facilitated.

(89.) Let B be any number (except unity) and supposed to be the base of a system of logarithms. By what has been already explained, any other number n is to be considered as a power of B, the exponent of which is the logarithm of n, and is written thus, ln. Thus we have the equation

$$n = B^{ln}.$$

If $n', n'' \cdots\cdot$ be any other numbers, we have, in like manner,

$$n' = B^{ln'}, \quad n'' = B^{ln''} \cdots\cdot$$

By multiplying these, we obtain

$$nn'n'' \cdots\cdot = B^{ln+ln'+ln''\cdots\cdot}$$

But by the definition of logarithms,

$$nn'n'' \cdots\cdot = B^{l(nn'n''\cdots\cdot)},$$

$$\because l(nn'n'' \cdots\cdot) = ln + ln' + ln'' \cdots\cdot$$

which shows that "*the logarithm of the product of any numbers is equal to the sum of their logarithms.*"

Again, let one of the equations be divided by another,

$$\frac{n}{n'} = B^{ln-ln'},$$

But also,

$$\frac{n}{n'} = B^{l\left(\frac{n}{n'}\right)},$$

$$\because l\left(\frac{n}{n'}\right) = ln - ln'.$$

That is, "*the logarithm of the quote of two numbers is found by subtracting the logarithm of the divisor from the logarithm of the dividend.*"

Thus, in general,

$$l\,\frac{nn'n''n''' \cdots\cdot}{mm'm''m''' \cdots\cdot} = (ln + ln' + ln'' \cdots\cdot) - (lm + lm' + lm'' \cdots\cdot).$$

By the assistance of logarithmic tables, multiplication may

therefore be thus performed. "*By the tables, find the logarithms of the several factors, and add them together, and then by the tables, find the number of which their sum is the logarithm; this number will be the product sought.*"

Or more generally, to determine the quote of any two products: "*By the tables, find the logarithms of all the factors of the dividend and of the divisor, and subtract the sum of the latter from the sum of the former, and find by the tables the number of which the remainder is the logarithm; that number is the quote sought.*"

It is plain that the logarithm of unity, whatever be the base, must $= 0$.

(90.) If both sides of the equation

$$n = \text{B}^{ln}$$

be raised to the mth power, and have their mth roots taken, we shall obtain

$$n^m = \text{B}^{mln}, \quad \sqrt[m]{n} = \text{B}^{\frac{ln}{m}}.$$

But also,

$$n^m = \text{B}^{l(n^m)}, \quad \sqrt[m]{n} = \text{B}^{l(\sqrt[m]{n})}.$$

Hence

$$l(n^m) = mln, \quad l\sqrt[m]{n} = \frac{ln}{m}.$$

Whence we have the following rule: "*The logarithm of any power of a number is obtained by multiplying the logarithm of the number by the exponent of the power; and the logarithm of any root of a number is obtained by dividing the logarithm of the number by the exponent of the root.*"

Hence, if a power or root of any number enter a formula which is to be computed, let the logarithm of the number be found from the tables, and let it be multiplied or divided by the exponent of the power or root; let the number of which the resulting product or quote is the logarithm be

likewise found by the tables, and this will be the power or root required.

These observations contain all the properties of logarithms necessary to render the elementary part of the theory of this science intelligible to the student. There are various particulars to be attended to in the use of tables, which, being of a merely practical and technical nature, are fully explained in the introductions to the usual collections of tables, and which it would be needless to enter upon here *.

SECTION VI.

Geometrical applications of plane trigonometry †.

PROP. XXVI.

(91.) *Given the base and vertical angle of a triangle, to construct it, so that the sum of its sides shall be a maximum.*

If the triangle be supposed to be circumscribed by a circle, the segment, whose base is the given side, will be given. The other sides of the triangle will be double the sines of half the arcs which they cut off, ∵ half their sum will be the sum of the sines of the halves of these arcs. Hence the question is reduced to this: given the sum of two arcs, to determine when the sum of their sines is a maximum. By (52.) [17],

$$\sin.A + \sin.B = 2\sin.\tfrac{1}{2}(A + B)\cos.\tfrac{1}{2}(A - B),$$

* For the means of computing the logarithms of numbers relatively to any given base, see Differential Calculus (68.), *et seq.*

† In Table V. the student will find various values for the side of a plane triangle, and the sine, cosine, and tangent of an angle in terms of the other sides and angles.

which, since $\frac{1}{2}(A+B)$ is given, is a maximum when $\cos.\frac{1}{2}(A-B)$ is a maximum, that is (36), when $(A - B) = 0 \because A = B$.

Hence the sum of the sides is a maximum when the triangle is isosceles.

PROP. XXVII.

(92.) *Given two sides of a triangle and the difference of the opposite angles, to compute the angles and the remaining sides.*

Let a and b be the two given sides. By (81.) we have

$$\cot.\tfrac{1}{2}C = \frac{a+b}{a-b}\tan.\tfrac{1}{2}(A - B),$$

by which C may be computed, and thence the other parts.

PROP. XXVIII.

(93.) *Given two angles and the sum or difference of the opposite sides, to compute the sides severally.*

Let A and B be the angles. By (73.),

$$\frac{\tan.\frac{1}{2}(A-B)}{\tan.\frac{1}{2}(A+B)}=\frac{a-b}{a+b},$$

by which, when either $a + b$ or $a - b$ is given, the other may be computed.

PROP. XXIX.

(94.) *Given an angle, one of the including sides and the difference of the remaining sides, to determine the triangle.*

The data in this case are C, a, and $b - c$. Hence $s - c$ and $s - b$ are given. Whence by (78.) we find

$$\tan.\tfrac{1}{2}B = \tan.\tfrac{1}{2}C\,.\,\frac{s-c}{s-b}.$$

PROP. XXX.

(95.) *To compute the segments into which the bisector of the vertical angle divides the base of a given triangle.*

Let the bisected angle be C, and x and y the segments of the side c, and let ϕ be the angle under the bisector and base. By (72.),

$$\frac{a}{x}=\frac{\sin.\phi}{\sin.\frac{1}{2}C}=\frac{b}{y}, \therefore \frac{a}{b}=\frac{x}{y},$$

$$\therefore \frac{a-b}{a+b}=\frac{x-y}{x+y}=\frac{x-y}{c},$$

$$\therefore x-y=c\cdot\frac{a-b}{a+b}.$$

Hence by (73.),

$$x-y=c\cdot\frac{\tan.\frac{1}{2}(A-B)}{\tan.\frac{1}{2}(A+B)}.$$

Thus $x-y$ may be found either by knowing c and the angles A and B, or by c and the ratio of the sides a and b.

PROP. XXXI.

(96.) *Given an angle, the sum or difference of the sides containing it, and the opposite side, to compute the remaining angles.*

Let the given angle be C, the side c, and the given sum or difference $a \pm b$. The problem is immediately solved by either of the formulæ (74.),

$$\cos.\tfrac{1}{2}(A-B)=\frac{a+B}{c}\sin.\tfrac{1}{2}C,$$

$$\sin.\tfrac{1}{2}(A-B)=\frac{a-b}{c}\cos.\tfrac{1}{2}C.$$

Whence the difference of the remaining angles being found, they may be severally determined since their sum is known.

PROP. XXXII.

(97.) *Given an angle, one of the sides which include it, and the sum of the other two sides, to determine the remaining angles.*

The data in this case are C, a, and $b + c$. Hence s and $s - a$ are known. The problem is therefore solved by the formula (78.),

$$\cot.\tfrac{1}{2}B = \tan.\tfrac{1}{2}C \frac{s}{s-a}.$$

PROP. XXXIII.

(98.) *Given two sides of a triangle and the included angle, to compute the segments of this angle made by a perpendicular upon the third side.*

Let a, b, be the given sides, and C the included angle. Let ω and ω' be the segments sought. By [26], (53.),

$$-\cot.\tfrac{1}{2}(\omega - \omega')\cot.\tfrac{1}{2}(\omega + \omega') = \frac{\cos.\omega + \cos.\omega'}{\cos.\omega - \cos.\omega'}.$$

But $\cos.\omega' = \sin.A$, $\cos.\omega = \sin.B$, $\because$

$$\cot.\tfrac{1}{2}(\omega - \omega')\cot.\tfrac{1}{2}(\omega + \omega') = \frac{\sin.A + \sin.B}{\sin.A - \sin.B},$$

$$\because \cot.\tfrac{1}{2}(\omega - \omega')\cot.\tfrac{1}{2}(\omega + \omega') = \frac{a+b}{a-b}.$$

If the perpendicular fall within the base,

$$C = \omega + \omega';$$

and if it fall without it,

$$C = \omega - \omega'.$$

Hence, in either case, ω and ω' may be computed.

PROP. XXXIV.

(99.) *Given two sides of a triangle and the included angle, to compute the segments of this angle made by the line drawn from its vertex to the middle point of the third side.*

Let a and b be the given sides, and C the included angle.
Let ω, ω', be the segments sought.
If the bisector be d, we have (72.),

$$\frac{\sin.\omega}{\sin.A}=\frac{\frac{1}{2}c}{d}, \quad \frac{\sin.\omega'}{\sin.B}=\frac{\frac{1}{2}c}{d},$$

$$\therefore \frac{\sin.\omega}{\sin.A}=\frac{\sin.\omega'}{\sin.B},$$

$$\therefore \frac{\sin.\omega+\sin.\omega'}{\sin.\omega-\sin.\omega'}=\frac{\sin.A+\sin.B}{\sin.A-\sin.B},$$

$$\therefore \frac{\tan.\frac{1}{2}(\omega+\omega')}{\tan.\frac{1}{2}(\omega-\omega')}=\frac{a+b}{a-b}.$$

By which, as in the last prop., ω and ω', may be computed.

PROP. XXXV.

(100.) *Given the four sides of a quadrilateral whose angles are supplemental, to determine its area and its angles.*

Let a, b, c, d, be the sides of the figure, and D the diagonal, which is the common base of the angles included by the sides ab and cd. Let the angle included by ab be φ, and that included by cd, φ'. By (75.),

$$D^2 = a^2 + b^2 - 2ab\cos.\varphi,$$

$$D^2 = c^2 + d^2 - 2cd\cos.\varphi',$$

$$\therefore (a^2 + b^2) - (c^2 + d^2) = 2(ab + cd)\cos.\varphi \cdot\cdot\cdot\cdot [A],$$

observing that $\cos.\varphi = -\cos.\varphi'$.

Let L be the area of the quadrilateral. By (77.) the

area of the triangle included by ab is $\frac{1}{2}ab\sin.\varphi$; and that included by cd is $\frac{1}{2}cd\sin.\varphi' = \frac{1}{2}cd\sin.\varphi$. Hence

$$2L = (ab + cd)\sin.\varphi \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad [B].$$

Let [A] be added to, and subtracted from, $2(ab + cd)$, and the results are

$$(a + b)^2 - (c - d)^2 = 2(ab + cd)(1 + \cos.\varphi),$$
$$-(a - b)^2 + (c + d)^2 = 2(ab + cd)(1 - \cos.\varphi),$$

which, by the conditions,

$$2\cos.^2\tfrac{1}{2}\varphi = 1 + \cos.\varphi,$$
$$2\sin.^2\tfrac{1}{2}\varphi = 1 - \cos.\varphi,$$

are reduced to

$$(s - c)(s - d) = (ab + cd)\cos.^2\tfrac{1}{2}\varphi,$$
$$(s - a)(s - b) = (ab + cd)\sin.^2\tfrac{1}{2}\varphi,$$

where $s = \frac{1}{2}(a + b + c + d)$. Hence by multiplying these together (55.), we obtain

$$(s - a)(s - b)(s - c)(s - d) = \tfrac{1}{4}(ab + cd)^2\sin.^2\varphi,$$

which, with [B], gives

$$L = \sqrt{(s - a)(s - b)(s - c)(s - d)},$$

where the area is expressed in terms of the sides, and the formula adapted for logarithms.

The angle φ may be computed by dividing the same pair of formulæ one by the other, which gives

$$\tan.^2\tfrac{1}{2}\varphi = \frac{(s-a)(s-b)}{(s-c)(s-d)},$$

which is likewise adapted for logarithms; and in the same manner the other angles may be computed.

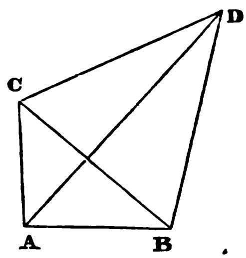
The analogy which these have to the results of (76.), (77.), for a triangle is evident. If $d = 0$, they become identical.

PROP. XXXVI.

(101.) *To determine the distances of two inaccessible, but visible objects, from the observer, and from each other.*

Let two convenient stations A, B, be selected, and the

distance between them measured. Let the angles CAD, CBD, DAB, and CBA, be observed, C and D being the proposed objects.



Hence all the angles about the points A and B may be determined, and thence the angles ACB and ADB, and hence the sides AC, BC, AD, BD, may be computed by (82.). We shall then have in either of the triangles CAD or CBD two sides, and the included angle to determine the base CD.

PART II.

SPHERICAL TRIGONOMETRY.

PART II.

SPHERICAL TRIGONOMETRY.

SECTION I.

Of the sphere.

(102.) If a semicircle be imagined to revolve round its diameter so as to move through 360°, its circumference will describe a surface which includes a solid called a *sphere.*

From this definition, it follows that all right lines drawn from the centre of the generating circle to the surface of the sphere are equal. Such lines are called *radii* of the sphere, and the point in which they meet is called its *centre.*

PROP. XXXVII.

(103.) *To determine the curve formed by the section of a spherical surface by a plane.*

1°. If the plane pass through the centre, it appears by (102.) that the section is a circle whose centre and radius are those of the sphere.

2°. If the plane do not pass through the centre, let a perpendicular be drawn from the centre to meet it, and let this be z, and the radius of the sphere r. The distance of any point in the section from the point where the perpendicular

z meets it is $\sqrt{r^2 - z^2}$, and this being the same for all points, the section is a circle whose centre is the foot of the perpendicular, and whose radius is $\sqrt{r^2 - z^2}$.

(104.) *Cor.* 1. The greatest section of a spherical surface is that whose plane passes through the centre, and all such sections are equal.

(105.) *Def.* Such sections are called *great circles.*

(106.) *Def.* Sections whose planes do not pass through the centre are called *lesser circles.*

(107.) *Cor.* 2. Lesser circles equidistant from the centre of the sphere are equal, and, in general, the circle diminishes as its plane recedes from the centre of the sphere.

PROP. XXXVIII.

(108.) *To draw a tangent plane to any point upon a sphere.*

Through the proposed point let a radius be drawn, and a plane perpendicular to the radius will be the tangent plane. For any other line drawn from the centre of the sphere to meet this plane is necessarily greater than the radius of the sphere which is the perpendicular upon it from the centre. Therefore the point where every such line meets the plane must be outside the sphere.

(109.) *Cor.* Tangent planes through the extremities of the same diameter are parallel, and *vice versâ;* and more than one tangent plane cannot be drawn to the same point.

PROP. XXXIX.

(110.) *To determine the locus of the centres of a system of parallel circles on a sphere.*

Let the diameter of the sphere be drawn which is per-

pendicular to their planes: this meets the plane of each circle at its centre (103.), and is therefore the locus sought.

(111.) *Def.* A diameter of the sphere is called the *axis* of those circles to whose planes it is perpendicular.

(112.) *Def.* The extremities of the axis of a circle are called its *poles.*

(113.) *Def.* Great circles whose planes are at right angles to the plane of any circle are called its *secondaries.*

(114.) *Cor:* 1. The planes of all secondaries pass through the axis, and their circumferences through the poles of their primary. The poles of a great circle may therefore always be determined by the intersections of any two secondaries.

(115.) *Cor.* 2. The arcs of all secondaries intercepted between the primary and its poles are $= 90^\circ$. For they subtend right angles at the centre of the secondaries.

(116.) *Cor.* 3. If three great circles be mutually secondary, the intersection of the planes of any two will be the axis of the third, and the intersections of the circumferences of any two will be the poles of the third.

PROP. XL.

(117.) *All great circles bisect each other, and a secondary bisects all parallels to its primary.*

1°. The intersection of the planes of two great circles is necessarily a diameter of the sphere, and a common diameter of both circles. Hence they necessarily bisect each other.

2°. By (114.) the plane of a secondary passes through the axis of its primary, and therefore through the centres (110) of all parallels to the primary. Hence it bisects all parallels.

PROP. XLI.

(118.) *To determine the relation between the radius of a lesser circle, its distance from the great circle to which it is parallel, and the arc of a secondary intercepted between them.*

Let z be the perpendicular distance between the plane of the lesser circle, and that of the great circle to which it is parallel, and let r be the radius of the lesser circle, R that of the sphere, and φ the arc of the secondary.

By (103.),

$$R^2 = z^2 + r^2.$$

Since R, r, and z, form a right angled triangle, the angle opposite the side z being equal to the arc φ *; if the radius (R) of the sphere be taken as unity, we have $r = \cos.\varphi$, $z = \sin.\varphi$.

PROP. XLII.

(119.) *Two secondaries intercept similar arcs of parallels to their primary, and these arcs are as the cosines of the arcs of the secondaries between the parallels and the primary.*

For the arcs of the parallels subtend at their respective centres angles equal to the inclinations of the planes of the secondaries, and these arcs being therefore similar, are as their radii, which are (118.) the cosines of the intercepted arcs of the secondaries.

* When an angle is said to be equal to an arc, it is meant that they contain the same number of degrees and parts of a degree.

PROP. XLIII.

(120.) *The angle under two great circles is equal to the angle under their planes.*

By the angle under two curves is meant the angle under two tangents to them drawn from their point of intersection; and by the angle under two planes is meant the angle under two right lines in those planes drawn from the same point in their line of intersection and perpendicular to that line.

Now the intersection of the planes of two great circles is the common diameter joining the points of intersection of their circumferences. Tangents to the circles drawn from these points of intersection are necessarily in the planes of the circles and perpendicular to their common diameter; hence the angle under these tangents is at the same time the angle under the circles and the angle under their planes.

(121.) *Cor.* Hence, if two great circles intersect, the angles vertically opposite are equal, as also the angles at the two points of intersection.

(122.) Distances on the surface of a sphere are measured by the arcs of great circles. One reason for this is, that the shortest line which can be drawn upon the surface of a sphere between any two points is the arc of a great circle joining them *.

(123.) If two points on the surface of a sphere be not at the opposite extremities of the same diameter, but one great circle can be drawn through them; for the centre of the sphere not lying *in directum* with them, will be sufficient to determine the plane of the great circle passing through

* Differential and Integral Calculus (483.).

them. In this case the points will be united in one direction by an arc less than a semicircle, and in the other by an arc greater than the semicircle. The one will be the shortest circular arc which can be drawn upon the surface of the sphere connecting the points, and the other the longest.

If, however, the two points be at opposite extremities of the same diameter, all planes passing through them must necessarily also pass through the centre of the sphere, and therefore all circles uniting them will be great circles. In this case the arcs of all circles uniting them will be semicircles.

PROP. XLIV.

(124.) *The angle under two great circles is equal to the distance between their poles.*

The axes of the great circles being perpendicular to their planes are inclined at the same angle as the planes of the circles. But the angle under the axes is obviously measured by the arc which joins their extremities, that is, by the distance between the poles of the great circles.

It should be observed, that the planes of the two great circles in general make an acute and obtuse angle, which are supplemental. Also, the pole of either is distant from the two poles of the other by supplemental arcs. The lesser arc is equal to the acute angle under the planes of the circles, and the greater arc to the obtuse angle.

PROP. XLV.

(125.) *The angle under two great circles is equal to the arc of a common secondary intercepted between them.*

For since this secondary passes through the poles of both, taking away from the equal quadrants of the secondary

between each circle and its pole, the common arc intercepted between one circle and the pole of the other, the remainders are the intercept of the common secondary between the two circles and the distance between their poles, which are therefore equal. But the latter is equal to the angle under the two circles.

SECTION II.

Of spherical triangles.

(126.) *Def.* Three points upon the surface of a sphere being connected by arcs of great circles, the figure formed on the surface by these arcs is called a *spherical triangle.*

When a great circle is drawn through two points, they may be considered connected by either segment into which they divide the circumference. One of the segments will in general be greater, and the other less than a semicircle. It is usual, however, to confine our attention to that segment which is less than a semicircle, and, accordingly, we always consider the sides of spherical triangles less than 180°.

PROP. XLVI.

(127.) *Any two sides of a spherical triangle taken together are greater than the third side.*

For if radii of the sphere be drawn to the three angles, they will form at the centre a solid angle bounded by three plane angles which are equal to the sides of the proposed triangle. It is proved (Eucl. lib. xi. prop. 20), that any two angles forming such a solid angle must be together greater than the third.

(128.) *Cor.* Hence the difference of any two sides of a spherical triangle is less than the third side.

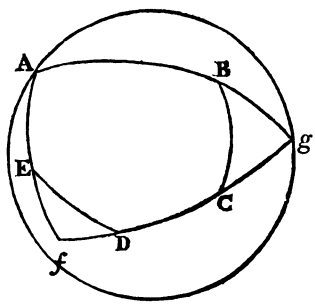
PROP. XLVII.

(129.) *The sum of the three sides of a spherical triangle is less than the circumference of a great circle.*

For let any two of the sides a, b, be produced through the third side c until they meet again. The produced parts $\pi - a$ and $\pi - b$ will, with the third side c, form a triangle. Hence by the last proposition,

$$\pi - a + \varpi - b > c,$$
$$\because 2\pi > a + b + c.$$

(130.) *Cor.* Hence the sum of the sides of a spherical polygon must be always less than the circumference of a great circle.



Let ABCDE be the polygon, and let AE, AB, and DC be produced so as to form the triangle Afg. By (129.),

$$\text{A}f + fg + \text{A}g < 2\pi;$$

and by (127.),

$$\text{E}f + \text{D}f > \text{DE},$$
$$\text{B}g + \text{C}g > \text{BC},$$

$$\because \text{A}f + fg + \text{A}g > \text{AE} + \text{ED} + \text{DC} + \text{CB} + \text{BA},$$
$$\text{AE} + \text{ED} + \text{DC} + \text{CB} + \text{BA} < 2\pi.$$

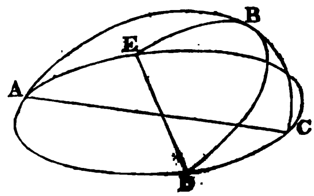
PROP. XLVIII.

(131.) *If three great circles intersect, they will divide the entire sphere into eight spherical triangles. Each of the hemispheres into which any one of the circles divides the sphere will be divided into four spherical triangles, which*

will be respectively equal side for side and angle for angle with the four triangles of the other hemisphere. Every two of these triangles have an angle in one equal to an angle in the other, and the sides opposite to these angles respectively equal, while the remaining sides and angles are supplemental.

This will be easily perceived by considering that the three circles mutually bisect each other.

Let AECD be one of the great circles and the base of one of the hemispheres. The four triangles are ABE, CBE, ABD, and CBD. Since ABC, AEC, EBD, ADC, are semicircles,



$$AB = \pi - BC,$$
$$BE = \pi - BD,$$
$$AE = \pi - CE,$$
$$AD = \pi - CD.$$

Also, since the angle BEC = BDC ∵ the angles BEA and BDC are supplemental; and the same observation may be applied to the other angles.

(132.) *Cor.* 1. Hence any spherical triangle being given, another may be found by changing two sides and the angles opposite them into their supplements; the remaining angles and sides continuing unchanged: that is, if

$$a, \quad b, \quad c,$$
$$A, \quad B, \quad C,$$

be the sides and angles of a spherical triangle,

$$\pi - a, \quad \pi - b, \quad c$$
$$\pi - A, \quad \pi - B, \quad C,$$

will be the sides and angles of another spherical triangle.

(133.) *Cor.* 2. Any four of the eight trian it

each from the others, must be together equal to an hemisphere.

PROP. XLIX.

(134.) *If the intersections of three great circles be the poles of three others, the intersections of the latter will be the poles of the former.*

Let a, b, c, be three great circles, and a', b', c', three others. The intersections of a and b are the poles of c', $\because$ the intersection of the planes of a and b is the axis of c' (114.), and is perpendicular to every line in the plane of c'. Therefore it is perpendicular to the intersection of the planes of a' and c'.

Also, for the same reason the intersection of the planes of b and c is perpendicular to the intersection of the planes of c' and a'. Since then the intersection of the planes of c' and a' is perpendicular to the intersection of the planes of b and c, and also to that of the planes b and a, it is perpendicular to two lines in the plane b, and is therefore perpendicular to the plane b itself. Eucl. lib. xi. Hence the intersection of the planes a' and c' is the axis of b, and the intersections of the circles a' and c' are therefore the poles of b.

In like manner it may be proved that the intersections of b' and c' are the poles of a, and those of a' and b' the poles of c.

PROP. L.

(135.) *If three great circles be drawn through the poles of the sides of a given spherical triangle, of any four triangles into which they divide the same hemisphere, three will have two sides equal to each pair of angles of the given triangle, and*

the angles opposed to these sides equal to the sides of the given triangle opposed to the former angles, the remaining side and angle being respectively supplemental to the remaining angle and side of the given triangle; and the fourth spherical triangle of the hemisphere will have sides which are the supplements of the angles and angles which are the supplements of the sides of the given triangle.

Let

$$a, \quad b, \quad c,$$
$$A, \quad B, \quad C,$$

be the sides and angles of the given triangle. The sides of the four triangles into which the same hemisphere is divided by great circles passing through the poles of a, b, c, are

1. $A, \quad B, \quad \pi - C,$
2. $B, \quad C, \quad \pi - A,$
3. $C, \quad A, \quad \pi - B,$
4. $\pi - A, \quad \pi - B, \quad \pi - C;$

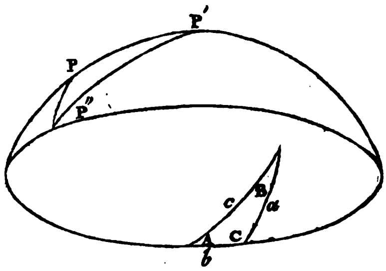
and the angles of these triangles opposed to the sides respectively are

1. $a, \quad b, \quad \pi - c,$
2. $b, \quad c, \quad \pi - a,$
3. $c, \quad a, \quad \pi - b,$
4. $\pi - a, \quad \pi - b, \quad \pi - c.$

These follow as consequences from the principle that the angle under two great circles is equal or supplemental to the distance between their poles. Any one of the four cases being proved, the other three may be deduced from it by (132.).

Let the triangle be that represented here, and let P be a pole of c, P' of b, and P'' of a.

It is evident that P and P' being the extremities of the axes of c and b, $PP' = A$. For the same reason $PP'' = B$ and $P'P'' = \pi - C$.



Thus the three sides of the triangle $PP'P''$ are

$$A, \quad B, \quad \pi - C,$$

These are the sides of the first of the four triangles. The sides of the other three follow by (131.).

The angles of $PP'P''$ may be derived from the sides a, b, c, by considering that the three vertices of a, b, c, must be the poles of the sides of $PP'P''$; and by the principle already established, we derive the values of the angles of $PP'P''$.

The angles of $PP'P''$ being obtained, the angles of the other three triangles follow by (132.).

(136.) *Def.* The triangles formed by great circles through the poles of a given triangle are called its *polar triangles;* and that whose sides are all supplemental to the angles of the given triangle is called its *supplemental polar triangle.*

(137.) *Def.* A spherical triangle, one of whose sides $= 90°$, is called a *quadrantal triangle.*

(138.) *Cor.* 1. If a spherical triangle be right angled, all its polar triangles will be quadrantal, and *vice versâ.*

(139.) *Cor.* 2. Let S be half the sum of the three angles of a triangle.

$$2S = A + B + C,$$
$$\because 2(S - A) = B + C - A,$$
$$2(S - B) = A + C - B,$$
$$2(S - C) = A + B - C.$$

The semisums of the sides of each of the four polar triangles respectively are

$$1. \ \tfrac{1}{2}(\pi + A + B - C) = \frac{\pi}{2} + (S - C),$$

$$2.\ \tfrac{1}{2}(\pi + B + C - A) = \frac{\pi}{2} + (S - A),$$

$$3.\ \tfrac{1}{2}(\pi + A + C - B) = \frac{\pi}{2} + (S - B),$$

$$4.\ \tfrac{1}{2}(3\pi - A - B - C) = \frac{3\pi}{2} - S.$$

The last being the supplemental polar triangle.

PROP. LI.

(140.) *To determine the limits of the magnitude of the sum of the three angles of a spherical triangle.*

It has been already proved that the major limit of the sum of the sides of a spherical triangle is 2π (129.).

Hence, if S be half the sum of the three angles, the major limit of

$$\frac{3\pi}{2} - S$$

is π, since this is half the sum of the sides of the supplemental polar triangle. Hence the minor limit of S is $\frac{\pi}{2}$, and therefore the minor limit of 2S is π.

Since the sum of the sides of a triangle has no minor limit, the major limit of S is given by the equation

$$\frac{3\pi}{2} - S = 0,$$

$$\therefore S = \frac{3\pi}{2}.$$

The major limit of 2S is therefore 3π.

Hence "*the sum of the three angles of a spherical triangle cannot be less than two right angles, nor greater than six.*"

Thus it appears that the sum of the angles of a spherical triangle is not, like that of plane triangles, constant, but varies

between the limits 180° and 540°, without being capable of becoming equal to either.

The supposition that no spherical triangle has sides greater than 180° necessarily infers by the polar triangles, that no spherical triangle can have an angle greater than 180°. There exist, however, spherical triangles which have angles as well as sides exceeding 180°; but as the values of their sides and angles can always be derived from the consideration of triangles whose sides and angles are less than 180°; the definition of spherical triangles is generally restricted to the latter, which contributes much to the simplicity of the investigations in spherical trigonometry.

(141.) *Cor.* 1. Hence a spherical triangle may have two or three angles, right or obtuse.

(142.) *Cor.* 2. If two angles be right, the sides opposite to them must be quadrants. For in that case, these sides are secondaries to the third side, and ∵ intersect in its poles. Hence the sides must each = 90°.

(143.) *Cor.* 3. If the three angles be right, the three sides will be quadrants, and the three great circles, if completed, will divide the surface of the sphere into eight equal equilateral quadrantal triangles.

PROP. LII.

(144.) *The angles at the base of an isosceles spherical triangle are equal, and vice versâ.*

From the centre of the sphere through the vertex of the isosceles triangle let a radius be drawn and produced. Through the extremities of the base let tangents be drawn to the sides until they meet the radius produced through the vertex. The intercepts of the produced radius between

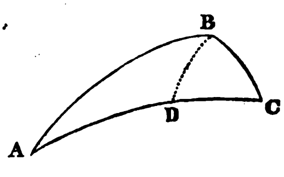
these tangents and the centre must be equal, since they are secants of the equal sides, and therefore the two tangents must meet the produced radius at the same point, and must themselves be equal, since they are the tangents of the equal sides.

Let tangents to base be drawn through its extremities and continued until they meet. They will also be equal. Let the point where they meet be connected with the point where the tangents to the sides meet, and two triangles will be thus formed, having the joining line as a common side, and the equal tangents to the base and side for the remaining sides. The angles therefore included by the tangents to the sides and base will be equal; but these are the angles of the spherical triangle at the base.

The converse of this proposition may immediately be inferred by the polar triangles. If two angles of the given triangle be equal, two sides of the corresponding polar triangle being equal to them, must also be equal, ∵ the angles opposite to these sides are equal; but these angles are equal to the sides of the given triangle which are opposite to the equal angles.

PROP. LIII.

(145.) *In a spherical triangle the greater side is opposed to the greater angle.*



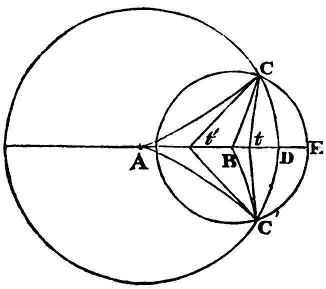
Let the angle ABC > ACB, and let DBC = DCB.

Hence by (144.) DC = BD.

But AD + BD > AB, ∵ AC > AB.

PROP. LIV.

(146.) *On the same base two, and only two, spherical triangles can be constructed whose conterminous sides are equal; and the vertices of these are similarly placed at opposite sides of the common base; and the angles of the two triangles opposite to the sides respectively equal, are equal.*



Let AB be the common base. At a distance AD from A, equal to the common length of the sides supposed to be terminated at A, let a lesser circle be described with A as pole. This circle must be the locus of the vertices of all triangles on the common base AB, and having their sides terminated at A of the common length.

In like manner let a lesser circle be described with B as pole, and at a distance BE equal to the common length of the sides terminated at B. The vertices of the triangles constructed on AB having their sides terminated at A equal to AD, and those terminated at B equal to BE must be at the same time on both circles, and must be their points of intersection. These can be but two, C, C′, similarly placed at opposite sides of the base AB.

To prove the equality of the angles C, C′, opposite to the common side AB let tangents be drawn to the sides CB, C′B, and CA, C′A; and let the points where they meet the plane of the circle AB be t, t', which being joined by a right line tt', the two plane triangles tct' and $tc't'$ will be respectively equilateral, and ∵ equiangular, ∵ the angle tct' is equal to

the angle $tc't'$; *i.e.* the angle ACB is equal to the angle AC′B.

In like manner, by supposing the side BC′ placed upon BC, the triangles lying at opposite sides, the angle BAC may be proved to be equal to BAC′; and in a similar way CBA may be proved to be equal to C′BA.

(147.) Although the sides and angles of these two triangles be respectively equal each to each, and, as will hereafter appear, their areas be also equal, yet their equality is not of that kind which allows of superposition. For if, while two angular points A and B of the triangles coincide, the third angular point C′ were brought to coincidence with C, the triangle ABC′ would be no longer upon the surface of the same sphere. The concavity of each of the sides ABC would be presented towards the concavity of each of the sides ABC′, and the concave surfaces of the two triangles would be opposed. The surface of the sphere of which the triangle ABC′ in its new position would be a part might be thus determined. From the centre of the sphere ABC draw a line perpendicular to the plane of the points A, B, and C; and let this perpendicular be produced beyond that plane until the produced part is equal to the perpendicular itself. With the extremity of the produced part as centre, and a radius equal to that of the original sphere, let another sphere be described: the triangle ABC′ in its new position will be on the surface of that sphere, as is obvious.

It is plain that the two spheres intersect each other in the lesser circle which circumscribes the triangle ABC.

(148.) Two triangles which are equal in the manner of ABC and ABC′ are said to be *symmetrically equal.* When they are equal so as to admit superposition, they are said to be *absolutely equal* *.

* LEGENDRE appears to have been the first to point out the distinction between absolute and symmetrical equality.

(149.) It follows from what has been observed, that two triangles on the surfaces of equal spheres, formed by joining three points in the lesser circle in which they intersect, will be *symmetrically equal.*

(150.) It is plain, that if the triangles be isosceles AC = BC = AC′ = BC′, they will be absolutely equal. For by interchanging the places of A and B, C′ will be brought upon C.

PROP. LV.

(151.) *Two spherical triangles which are mutually equilateral*, are also mutually equiangular, the equal angles being opposed to the equal sides.*

The triangles being supposed to be placed so that two sides respectively equal shall coincide, must either be placed at the same or opposite sides of those coincident sides. If they be at the same side, the remaining vertical points must coincide by (146.), and therefore the sides themselves (123.); and therefore the triangles are *absolutely* equal, the equal angles be opposed to the equal sides.

If they be placed at opposite sides of the coincident sides, they will be *symmetrically equal,* and the proposition has been already proved.

(152.) *Cor.* Hence the arc drawn from the vertex of an isosceles spherical triangle to the point of bisection of the base, bisects the vertical angle, and is perpendicular to the base.

---

* In comparing the sides of spherical triangles, their magnitudes are measured by the number of degrees they contain. In this sense the above proposition, and those which succeed it, may easily be extended to triangles upon the surfaces of different spheres, although the demonstrations of some of them suppose the triangles to be upon the same spherical surface.

PROP. LVI.

(153.) *If two spherical triangles be mutually equiangular, they will also be mutually equilateral, the equal sides being opposed to the equal angles.*

For since the angles are equal, the sides of their supplemental polar triangles are equal (136.); therefore their angles are equal (151.) But these angles are the supplements of the sides of the proposed triangles.

(154.) Most of the criterions for the determination of the equality of spherical triangles are the same as those for plane or rectilinear triangles. The above is, however, an exception. If the three angles of two plane triangles be equal, the sides will be proportional, but not necessarily equal. If the spherical triangles be described upon the surfaces of the same or equal spheres, their proportionality necessarily infers their equality, since they are related to the same or equal radii. If, however, they be described upon the surfaces of unequal spheres, the equality which follows from the preceding demonstration is only an equality as to the number of degrees in the sides, which amounts to no more than proportionality as to absolute length, the sides of the triangles being proportional to the radii of the spheres.

PROP. LVII.

(155.) *If two spherical triangles have an angle, and the sides which contain it equal each to each, the remaining side and angles will also be respectively equal.*

For one of the triangles admits of the superposition described in Euc. lib. i. prop. 4. either upon the other or upon one which is symmetrically equal to the other. Hence the

two proposed triangles are either absolutely or symmetrically equal.

(156.) *Cor.* 1. Hence the arc which bisects the vertical angle of an isosceles triangle will bisect its base, and be perpendicular to it.

(157.) *Cor.* 2. Also, if the arc from the vertex of a spherical triangle perpendicular to the base bisects the base, the triangle will be isosceles.

PROP. LVIII.

(158.) *If two spherical triangles have a side, and the angles between which it is placed respectively equal, the remaining angle and sides are also equal each to each.*

For they have two polar triangles which have sides equal to the two angles which are given, equal each to each, and the included angles the supplements of the sides which are given equal. Hence the remaining sides and angles of these polar triangles must be equal, each to each (155.) But the remaining sides are the supplements of the remaining angles of the proposed triangles, and the remaining angles are respectively equal to the remaining sides of the proposed triangles.

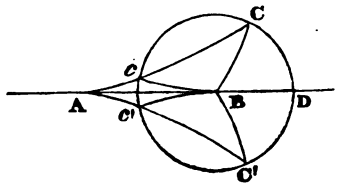
(159.) *Cor.* If the arc which bisects the vertical angle of a spherical triangle be secondary to the base, the triangle is isosceles.

PROP. LIX.

(160.) *If two spherical triangles have two sides respectively equal, each to each, and the angles opposed to one pair of equal sides equal, the angles opposed to the other pair of equal sides must be either equal or supplemental.*

Let the vertices of the equal angles and vertices of the

angles included by the sides respectively equal, be conceived to be placed the one upon the other, so that the sides AB which include the equal angles at A shall coincide.



The two triangles will then be placed either at the same or opposite sides of AB. In either case, the vertices of the angles opposed to AB must be placed upon a lesser circle described round the pole B at a distance BD equal to the equal sides which are opposed to the equal angles A.

If the triangles be at the same side of AB, the sides which include the equal angles A must coincide, and the vertices of the angles opposite to AB must either coincide at C or *c*, or one must be at C and the other at *c*. In the first case the angles opposed to AB are equal. In the second case, since the triangle *c*BC is isosceles, the angles B*c*C and BC*c* are equal, and therefore the angles opposed to AB are supplemental.

If the triangles lie at opposite sides of AB, the vertices opposed to AB must be found in either of four positions,

1. *cc'*, 2. CC',
3. *c*C', 4. *c'*C.

In the first two cases, the three sides of the triangles are respectively equal, and therefore the angles opposed to AB are equal. In the last two the angles are proved to be supplemental by the isosceles triangles CB*c* and C'B*c'*, as before.

(161.) *Cor.* 1. Hence if the angles opposed to the side AB be of the same affection, *i. e.* both acute or both obtuse, they must be equal, and the triangles will be necessarily either absolutely or symmetrically equal.

(162.) *Cor.* 2. If one of the angles opposed to the side AB be right, the other must be also right, and therefore the triangles either absolutely or symmetrically equal.

PROP. LX.

(163.) *If two spherical triangles have two angles and the sides opposite to one pair of them respectively equal, the sides opposite to the other pair must be either equal or supplemental.*

For there are two polar triangles whose sides are equal to the proposed angles, and the angles opposed to these sides are equal respectively to the sides whose equality is given, and to those which are affirmed to be equal or supplemental (135.). Now, by the last proposition, the angles of the polar triangles opposed to the given equal sides are either equal or supplemental, and therefore the corresponding sides of the proposed triangles are supplemental or equal.

(164.) *Cor.* 1. If the remaining pair of sides be of the same affection, they must be equal, and the triangles will be either absolutely or symmetrically equal.

(165.) *Cor.* 2. If either of the remaining sides be a quadrant, the other must be also a quadrant, and the triangles will be either absolutely or symmetrically equal.

PROP. LXI.

(166.) *If two spherical triangles have two sides equal each to each, the remaining side in the one is greater or less than the remaining side in the other, according as the angle opposed to it is greater or less in the one than in the other.*

This is proved in the same manner as the corresponding proposition relative to plane triangles.

SECTION III.

Of the areas of spherical figures.

(167.) *Def.* Two semicircles being described upon the same diameter of a sphere, that part of the surface of the sphere which they include is called a *lune*.

(168.) *Def.* The common diameter of the semicircles is called the *axis* of the lune.

(169.) *Def.* The angle under the planes of the semicircles is called the *angle* of the lune.

PROP. LXII.

(170.) *Lunes of the same sphere, whose angles are equal, have equal surfaces.*

For if their axes be supposed to be placed in coincidence as well as the planes of two of their semicircles, the planes of the other two semicircles being turned in the same direction, must coincide, since the angles of the lunes are equal. Therefore the semicircles which bound the lunes will coincide, and therefore their surfaces will necessarily also coincide and be equal.

PROP. LXIII.

(171.) *Given the surface of the sphere and the angle of a lune, to determine its area.*

Let ω be the angle of the lune, and S the surface of the sphere. Let the angle ω be divided into any number of

parts n. It is plain that the lune may be divided into a number of (n) equal lunes; the angles of which will be $\frac{\omega}{n}$. Now as often as $\frac{\omega}{n}$ is contained in 360°, so often will one of these lunes be contained in the whole surface of the sphere. Hence it appears that the lune whose angle is ω, bears to the surface S the ratio $\omega : 360°$. If L be the lune

$$\text{L} = \text{S}\frac{\omega}{360^\circ} = \text{S}\cdot\frac{\omega}{2\pi}.$$

The surface of a sphere is proved to be equal to four times the area of one of its great circles *. Hence

$$\text{S} = 4r^2\pi,$$

$$\because \text{L} = 2r^2\omega.$$

(172.) *Cor.* 1. Lunes of the same sphere are as their angles.

(173.) *Cor.* 2. Lunes of different spheres are as the products of their angles and the squares of their axes.

(174.) *Cor.* 3. The whole surface of the sphere consists of four rectangular lunes.

(175.) *Cor.* 4. A secondary to the sides of a lune divides it into two equal rectangular isosceles triangles, whose vertical angles are those of the lune.

PROP. LXIV.

(176.) *Triangles which are mutually equilateral are equal in area.*

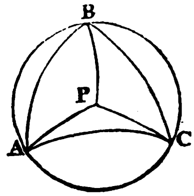
1. If they be absolutely equal, they admit superposition, and are therefore equal in area.

2. If they be symmetrically equal, let them be circumscribed by lesser circles. These lesser circles will be equal by (147.).

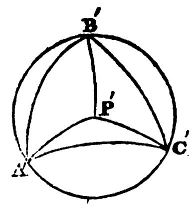
---

* Differential and Integral Calculus (276.).

Hence if their poles be connected with the three angles, the connecting lines PA, PB, PC, must be equal to each other, and to P'A', P'B', P'C'. Hence they divide each triangle into three isosceles triangles, whose sides are respectively equal, each to each, and these are therefore absolutely equal, and their areas are therefore equal; and, therefore, areas of the entire triangles are equal.



If the poles of the lesser circles fall outside the triangles, the area of each triangle is the difference between one of the isosceles triangles and the sum of the other two.



PROP. LXV.

(177.) *Given the surface of the sphere, to determine the area of a given triangle.*

Let each pair of sides of the triangle be produced through the extremities of the third side until they intersect. There will thus be three lunes formed, whose angles will be the three angles of the triangle, and their surfaces respectively will be

$$2r^2A, \qquad 2r^2B, \qquad 2r^2C,$$

r being the radius of the sphere, and A, B, C, the angles of the triangle (171.).

In the surfaces of these lunes the given triangle is three times repeated, and with it three of the eight triangles into which the whole sphere is divided by the circles which form the given triangle (131.) The surface of the hemisphere is equal to those three triangles together with the given one. Hence the sum of the three lunes exceeds the hemisphere by

twice the area of the given triangle. Therefore, if D be the area of the triangle

$$D = r^2(A + B + C) - r^2\pi,$$

$$\therefore D = r^2[(A + B + C) - \pi].$$

Hence the area of a spherical triangle is equal to that of a lune of the same sphere whose angle is equal to half the excess of the sum of the three angles of the triangle above two right angles.

In the formula just obtained, the angles A, B, C, π, are related to the radius unity (24.).

If they should be expressed in seconds, it will be necessary to divide each angle by 206265 (7.).

(178.) *Cor.* 1. The areas of triangles on the surface of the same sphere are proportional to the excess of the sums of their angles above two right angles.

(179.) *Cor.* 2. The areas of triangles upon different spherical surfaces are as the squares of the radii multiplied into the excesses of the sums of their angles above two right angles.

PROP. LXVI.

(180.) *To determine the surface of a spherical polygon.*

Let the polygon be resolved into triangles by diagonals drawn from any angle. The sum of the triangles will be equal in general to the polygon, and the sum of the angles of the triangles equal to the sum of the angles of the polygon. If the polygon have n sides, there will be $n - 2$ triangles; and if S be the sum of its angles and D its area,

$$D = r^2[S - (n - 2)\pi].$$

Hence the areas of different polygons on the same sphere are proportional to the excesses of the sums of their angles above two right angles multiplied by two less than the number of sides.

SECTION IV.

Trigonometrical formulæ expressing relations between the sides and angles of a spherical triangle.

(181.) The analytical formulæ which express all the various trigonometrical relations of the sides and angles of a spherical triangle, although very numerous, and many of them apparently unconnected, may, nevertheless, be all derived from one formula, which may therefore be considered as the foundation of the whole structure of spherical trigonometry. This formula therefore may be considered in spherical trigonometry in the same point of view as that for the sine of the sum of two arcs in plane trigonometry, and, as in that case, we shall establish it by geometrical construction, and subsequently derive all others from it.

Let a, b, c, be the sides, and A, B, C, the angles of a spherical triangle, as usual. From the vertex of the angle C let tangents be drawn to the arcs a and b; and from the centre of the sphere let the radii through the vertices of the angles A and B be drawn and produced to meet the tangents; and let the points where they meet the tangents be connected by a right line D. The produced radii are evidently the secants of the sides a, b, the radius of the sphere being unity, and the parts of the tangents intercepted between the secants and the vertex of C are the tangents of the sides. The angle under the tangents is equal to C, and that under the secants to c.

By applying the principle (75.) to the two plane triangles

of which D is a common side, and of which the remaining sides are the tangents and secants of a and b, we obtain

$$\text{D}^2 = \tan.^2 a + \tan.^2 b - 2\tan.a \tan.b \cos.\text{C},$$
$$\text{D}^2 = \sec.^2 a + \sec.^2 b - 2\sec.a \sec.b \cos.c.$$

Subtracting the former from the latter, and observing the conditions,

$$\sec.^2 a - \tan.^2 a = 1,$$
$$\sec.^2 b - \tan.^2 b = 1,$$
$$\sec.a \sec.b = \frac{1}{\cos.a \cos.b},$$
$$\tan.a \tan.b = \frac{\sin.a \sin.b}{\cos.a \cos.b},$$

we obtain

$$\cos.a \cos.b + \sin.a \sin.b \cos.\text{C} - \cos.c = 0.$$

This principle being successively applied to each of the three angles A, B, C, gives a system of three formulæ, which may be expressed thus:

$$\cos.\left|\begin{matrix} a \\ b \\ c \end{matrix}\right. - \cos.\left|\begin{matrix} \text{A} \\ \text{B} \\ \text{C} \end{matrix}\right. \sin.\left|\begin{matrix} b \\ c \\ a \end{matrix}\right. \sin.\left|\begin{matrix} c \\ a \\ b \end{matrix}\right. - \cos.\left|\begin{matrix} b \\ c \\ a \end{matrix}\right. \cos.\left|\begin{matrix} c \\ a \\ b \end{matrix}\right. = 0 \cdot \cdot \cdot \cdot \quad [1],$$

which formulæ are the foundation of spherical trigonometry.

We have adopted this method of grouping the formulæ in preference to the usual method of repeating cos., sin., &c. before each side and angle, because it exhibits more plainly the symmetry which prevails in the system, and impresses the formulæ more readily upon the memory.

(182.) By the equations

$$\cos.a - \cos.\text{A} \sin.b \sin.c - \cos.b \cos.c = 0,$$
$$\cos.(b \pm c) \pm \sin.b \sin.c - \cos.b \cos.c = 0,$$

we find

$$\cos.(b + c) - \cos.a + \sin.b \sin.c(1 + \cos.\text{A}) = 0,$$
$$\cos.(b - c) - \cos.a - \sin.b \sin.c(1 - \cos.\text{A}) = 0.$$

But

$$\cos.(b+c)-\cos.a=2\sin.\tfrac{1}{2}(a+b+c)\sin.\tfrac{1}{2}(b+c-a),$$
$$\cos.(b-c)-\cos.a=2\sin.\tfrac{1}{2}(a+b-c)\sin.\tfrac{1}{2}(a+c-b),$$
$$1+\cos.A=2\cos.^2\tfrac{1}{2}A,$$
$$1-\cos.A=2\sin.^2\tfrac{1}{2}A.$$

Hence, if

$$s=\tfrac{1}{2}(a+b+c),$$
$$\therefore (s-a)=\tfrac{1}{2}(b+c-a),$$
$$(s-b)=\tfrac{1}{2}(a+c-b),$$
$$(s-c)=\tfrac{1}{2}(a+b-c),$$

we obtain

$$\left.\begin{aligned} &\sin.b\sin.c\cos.^2\tfrac{1}{2}A=\sin.s\sin.(s-a)\\ &\sin.b\sin.c\sin.^2\tfrac{1}{2}A=\sin.(s-b)\sin.(s-c)\\ &\sin.^2b\sin.^2c\sin.^2A=4\sin.s\sin.(s-a)\sin.(s-b)\sin.(s-c)\\ &\tan.^2\tfrac{1}{2}A=\frac{\sin.(s-b)\sin.(s-c)}{\sin.s\sin.(s-a)}. \end{aligned}\right\}[2]$$

It is evident that four formulæ analogous to these are applicable to each of the three angles, and may be derived from these by merely changing the letters.

(183.) By the third of the group [2] we obtain

$$\sin.A=\frac{2[\sin.s\sin.(s-a)\sin.(s-b)\sin.(s-c)]^{\frac{1}{2}}}{\sin.b\sin.c},$$

from which we infer

$$\frac{\sin.A}{\sin.a}=\frac{2[\sin.s\sin.(s-a)\sin.(s-b)\sin.(s-c)]^{\frac{1}{2}}}{\sin.a\sin.b\sin.c}\quad\ldots[3].$$

This formula being a symmetrical function of the sides of the triangle will remain unchanged, if b be changed into a or c, and *vice versâ*. Hence we infer in general, that

$$\frac{\sin.A}{\sin.a}=\frac{\sin.B}{\sin.b}=\frac{\sin.C}{\sin.c}\quad\ldots\ldots\ldots[4];$$

each of these being equal to the same symmetrical function of the sides.

(184.) If by the first and third of the formulæ [1] $\cos.c$ be eliminated, we obtain

$$\cos.A \sin.c + \cos.C \sin.a \cos.b - \sin.b \cos.a = 0.$$

Eliminating $\sin.c$ by

$$\frac{\sin.C}{\sin.c} = \frac{\sin.A}{\sin.a},$$

we find

$$\cos.C \cos.b = \cot.a \sin.b - \cot.A \sin.C;$$

which being applied successively to each of the three angles, gives the system of formulæ

$$\cos.\left|\begin{matrix}A\\B\\C\end{matrix}\right.\cos.\left|\begin{matrix}c\\a\\b\end{matrix}\right.+\sin.\left|\begin{matrix}A\\B\\C\end{matrix}\right.\cot.\left|\begin{matrix}B\\C\\A\end{matrix}\right.-\sin.\left|\begin{matrix}c\\a\\b\end{matrix}\right.\cot.\left|\begin{matrix}b\\c\\a\end{matrix}\right. = 0 \quad . \quad . \quad . \quad [5].$$

(185.) By the first and second of [1] eliminating $\cos.a$, the result is

$$\sin.a \cos.B - \cos.b \sin.c + \sin.b \cos.c \cos.A = 0,$$

which applied to the angles successively, gives

$$\sin.\left|\begin{matrix}a\\b\\c\end{matrix}\right.\cos.\left|\begin{matrix}B\\C\\A\end{matrix}\right.-\cos.\left|\begin{matrix}b\\c\\a\end{matrix}\right.\sin.\left|\begin{matrix}c\\a\\b\end{matrix}\right.+\sin.\left|\begin{matrix}b\\c\\a\end{matrix}\right.\cos.\left|\begin{matrix}c\\a\\b\end{matrix}\right.\cos.\left|\begin{matrix}A\\B\\C\end{matrix}\right. = 0 \quad . \quad . \quad . \quad [6].$$

(186.) By the first and third of [1] we obtain

$$\cos.C \sin.a \sin.b = - \cos.a \cos.b + \cos.c,$$

$$\cos.A \sin.c \sin.b = - \cos.b \cos.c + \cos.a,$$

$$\because \cos.A \sin.c \cos.b \sin.b = - \cos.^2 b \cos.c + \cos.a \cos.b,$$

adding the first and last of these, and dividing the result by $\sin.b$, we have

$$\sin.a \cos.C - \cos.c \sin.b + \sin.c \cos.A \cos.b = 0,$$

which, being successively applied to the three angles, gives

$$\sin.\left|\begin{matrix}a\\b\\c\end{matrix}\right.\cos.\left|\begin{matrix}C\\A\\B\end{matrix}\right.-\cos.\left|\begin{matrix}c\\a\\b\end{matrix}\right.\sin.\left|\begin{matrix}b\\c\\a\end{matrix}\right.+\sin.\left|\begin{matrix}c\\a\\b\end{matrix}\right.\cos.\left|\begin{matrix}A\\B\\C\end{matrix}\right.\cos.\left|\begin{matrix}b\\c\\a\end{matrix}\right. = 0 \quad . \quad . \quad . \quad [7].$$

(187.) By adding and subtracting the third of [6] and [7], we obtain

$$(\cos.A + \cos.B)\sin.c - (1 - \cos.C)\sin.(a + b) = 0,$$

$$(\cos.A - \cos.B)\sin.c - (1 + \cos.C)\sin.(a - b) = 0,$$

$$\left.\begin{aligned}\therefore \sin.(a+b)\sin.^2\tfrac{1}{2}c-\cos.\tfrac{1}{2}(A-B)\cos.\tfrac{1}{2}(A+B)\sin.c=0\\ \sin.(a-b)\cos.^2\tfrac{1}{2}c-\sin.\tfrac{1}{2}(A-B)\sin.\tfrac{1}{2}(A+B)\sin.c=0\end{aligned}\right\}[8].$$

(188.) By [2] we obtain

$$\sin.^2\tfrac{1}{2}A = \frac{\sin.(s-b)\sin.(s-c)}{\sin.b\,\sin.c},$$

$$\cos.^2\tfrac{1}{2}A = \frac{\sin.s\,\sin.(s-a)}{\sin.b\,\sin.c},$$

$$\sin.^2\tfrac{1}{2}B = \frac{\sin.(s-a)\sin.(s-c)}{\sin.a\,\sin.c},$$

$$\cos.^2\tfrac{1}{2}B = \frac{\sin.s\,\sin.(s-b)}{\sin.a\,\sin.c},$$

which, with

$$\sin.\tfrac{1}{2}(A \pm B) = \sin.\tfrac{1}{2}A\cos.\tfrac{1}{2}B \pm \sin.\tfrac{1}{2}B\cos.\tfrac{1}{2}A,$$
$$\sin.a\sin.b\cos.^2\tfrac{1}{2}C = \sin.s\sin.(s-c),$$
$$\sin.(s-b)+\sin.(s-a) = 2\sin.\tfrac{1}{2}c\cos.\tfrac{1}{2}(a-b),$$
$$\sin.(s-b)-\sin.(s-a) = 2\cos.\tfrac{1}{2}c\sin.\tfrac{1}{2}(a-b),$$

give

$$\left.\begin{aligned}\sin.\tfrac{1}{2}(A+B) &= \frac{\cos.\tfrac{1}{2}C}{\cos.\tfrac{1}{2}c}\cos.\tfrac{1}{2}(a-b)\\ \sin.\tfrac{1}{2}(A-B) &= \frac{\cos.\tfrac{1}{2}C}{\sin.\tfrac{1}{2}c}\sin.\tfrac{1}{2}(a-b)\end{aligned}\right\}\quad\ldots\quad[9].$$

(189.) By a process precisely similar, we obtain

$$\left.\begin{aligned}\cos.\tfrac{1}{2}(A+B) &= \frac{\sin.\tfrac{1}{2}C}{\cos.\tfrac{1}{2}c}\cos.\tfrac{1}{2}(a+b)\\ \cos.\tfrac{1}{2}(A-B) &= \frac{\sin.\tfrac{1}{2}C}{\sin.\tfrac{1}{2}c}\sin.\tfrac{1}{2}(a+b)\end{aligned}\right\}\quad\ldots\quad[10].$$

(190.) By dividing the formulæ of [9] by those of [10] respectively, we obtain

$$\left.\begin{aligned}\tan.\tfrac{1}{2}(A+B) &= \frac{\cos.\tfrac{1}{2}(a-b)}{\cos.\tfrac{1}{2}(a+b)}\cot.\tfrac{1}{2}C\\ \tan.\tfrac{1}{2}(A-B) &= \frac{\sin.\tfrac{1}{2}(a-b)}{\sin.\tfrac{1}{2}(a+b)}\cot.\tfrac{1}{2}C\end{aligned}\right\}\quad\ldots\quad[11].$$

(191.) The polar triangles (135.) furnish a rule by which every group of formulæ expressing relations between the

sides and angles of a spherical triangle can be converted into another group giving other relations between the same quantities. This circumstance, without increasing the length of the analytical process, doubles the number of useful results. Any formula may be applied to the supplemental polar triangle by changing the sides into the supplements of the angles, and *vice versâ.* But since the sine and cosecant of the supplement of an angle are the same as those of the angle itself, and the cosine, tangent, cotangent, and secant of the supplement only differ from those of the angle itself in sign, it follows that it is allowed in any formulæ to change a, b, c, into A, B, C, and *vice versâ,* provided that the signs of all cosines, tangents, cotangents, and secants, be changed.

(192.) The other polar triangles also point out transformations of which formulæ are susceptible. They show that in any formula two sides may be changed into the opposite angles, and *vice versâ;* and the remaining side into the supplement of the angle opposed to it, and *vice versâ.* In other words, the characters a, b, c, may be changed into A, B, C, and *vice versâ;* the signs of the cosines, tangents, cotangents, and secants of any one of the sides and the angle opposed to it being changed; but the other quantities retaining their signs.

(193.) The changes thus indicated being effected upon all the formulæ which have been established in this section, the result will be a series of analogous formulæ.

It may be observed in general, that if a formula be a symmetrical function of the sides and angles, this change in the parts produces no change in the whole.

This observation applies to [4].

(194.) To make this transformation on [2], [3], [9], [10], and [11], let

$$S = \tfrac{1}{2}(A + B + C);$$

and let the sides and angles of the supplemental polar triangle be

$$a', \quad b', \quad c',$$
$$A', \quad B', \quad C'.$$

Hence by (135.),

$$a' + A = \pi, \quad b' + B = \pi, \quad c' + C = \pi,$$
$$A' + a = \pi, \quad B' + b = \pi, \quad C' + c = \pi.$$

Hence

$$\sin.a' = \sin.A, \qquad \sin.b' = \sin.B', \qquad \sin.c' = \sin.C,$$
$$\sin.\tfrac{1}{2}A' = \cos.\tfrac{1}{2}a, \quad \cos.\tfrac{1}{2}A' = \sin.\tfrac{1}{2}a, \quad \tan.\tfrac{1}{2}A' = \cot.\tfrac{1}{2}a,$$

$$s' = \frac{3\pi}{2} - S,$$

$$s' - a' = \frac{\pi}{2} - (S - A),$$

$$s' - b' = \frac{\pi}{2} - (S - B),$$

$$s' - c' = \frac{\pi}{2} - (S - C).$$

(195.) These observations being attended to, the changes already prescribed being made in the formulæ [1], [2], [3], [5], [6], [7], [8], [9], [10], and [11], we obtain the following systems of formulæ.

$$\cos.\left|\begin{matrix}A\\B\\C\end{matrix}\right. - \cos.\left|\begin{matrix}a\\b\\c\end{matrix}\right.\sin.\left|\begin{matrix}B\\C\\A\end{matrix}\right.\sin.\left|\begin{matrix}C\\A\\B\end{matrix}\right. + \cos.\left|\begin{matrix}B\\C\\A\end{matrix}\right.\cos.\left|\begin{matrix}C\\A\\B\end{matrix}\right. = 0 \quad \cdot \quad \cdot \quad \cdot \quad [12].$$

$$\left.\begin{matrix}\sin.B\sin.C\cos.^2\tfrac{1}{2}a = \cos.(S-B)\cos.(S-C)\\ \sin.B\sin.C\sin.^2\tfrac{1}{2}a = -\cos.S\cos.(S-A)\\ \sin^2B\sin^2C\sin^2a = -4\cos.S\cos(S-A)\cos(S-B)\cos(S-C)\\ \cot.^2\tfrac{1}{2}a = -\dfrac{\cos.(S-B)\cos.(S-C)}{\cos.S\cos.(S-A)}.\end{matrix}\right\}[13].$$

$$\frac{\sin.a}{\sin.A} = \frac{2[-\cos.S\cos(S-A)\cos.(S-B)\cos(S-C)]^{\frac{1}{2}}}{\sin.A\sin.B\sin.C} \quad \cdot \quad \cdot \quad [14].$$

$$\cos.\left|\begin{matrix}a\\b\\c\end{matrix}\right.\cos.\left|\begin{matrix}C\\A\\B\end{matrix}\right. - \sin.\left|\begin{matrix}a\\b\\c\end{matrix}\right.\cot.\left|\begin{matrix}b\\c\\a\end{matrix}\right. + \sin.\left|\begin{matrix}C\\A\\B\end{matrix}\right.\cot.\left|\begin{matrix}B\\C\\A\end{matrix}\right. = 0 \quad \cdot \quad \cdot \quad \cdot \quad [15].$$

$$\sin.\left|\begin{matrix}A\\B\\C\end{matrix}\right.\cos.\left|\begin{matrix}b\\c\\a\end{matrix}\right.-\cos.\left|\begin{matrix}B\\C\\A\end{matrix}\right.\sin.\left|\begin{matrix}C\\A\\B\end{matrix}\right.-\sin.\left|\begin{matrix}B\\C\\A\end{matrix}\right.\cos.\left|\begin{matrix}C\\A\\B\end{matrix}\right.\cos.\left|\begin{matrix}a\\b\\c\end{matrix}\right.=0\cdots[16].$$

$$\sin.\left|\begin{matrix}A\\B\\C\end{matrix}\right.\cos.\left|\begin{matrix}c\\a\\b\end{matrix}\right.-\cos.\left|\begin{matrix}C\\A\\B\end{matrix}\right.\sin.\left|\begin{matrix}B\\C\\A\end{matrix}\right.-\sin.\left|\begin{matrix}C\\A\\B\end{matrix}\right.\cos.\left|\begin{matrix}a\\b\\c\end{matrix}\right.\cos.\left|\begin{matrix}B\\C\\A\end{matrix}\right.=0\cdots[17].$$

$$\left.\begin{matrix}\sin.(A+B)\cos.^2\tfrac{1}{2}c-\cos.\tfrac{1}{2}(a-b)\cos.\tfrac{1}{2}(a+b)\sin.C=0\\ \sin.(A-B)\sin.^2\tfrac{1}{2}c-\sin.\tfrac{1}{2}(a-b)\sin.\tfrac{1}{2}(a+b)\sin.C=0\end{matrix}\right\}\cdots[18].$$

$$\left.\begin{aligned}\sin.\tfrac{1}{2}(a+b)&=\frac{\sin.\frac{1}{2}c}{\sin.\frac{1}{2}C}\cos.\tfrac{1}{2}(A-B)\\ \sin.\tfrac{1}{2}(a-b)&=\frac{\sin.\frac{1}{2}c}{\cos.\frac{1}{2}C}\sin.\tfrac{1}{2}(A-B)\end{aligned}\right\}\quad\cdots\quad[19].$$

$$\left.\begin{aligned}\cos.\tfrac{1}{2}(a+b)&=\frac{\cos.\frac{1}{2}c}{\sin.\frac{1}{2}C}\cos.\tfrac{1}{2}(A+B)\\ \cos.\tfrac{1}{2}(a-b)&=\frac{\cos.\frac{1}{2}c}{\cos.\frac{1}{2}C}\sin.\tfrac{1}{2}(A+B)\end{aligned}\right\}\quad\cdots\quad[20].$$

$$\left.\begin{aligned}\tan.\tfrac{1}{2}(a+b)&=\frac{\cos.\frac{1}{2}(A-B)}{\cos.\frac{1}{2}(A+B)}\tan.\tfrac{1}{2}c\\ \tan.\tfrac{1}{2}(a-b)&=\frac{\sin.\frac{1}{2}(A-B)}{\sin.\frac{1}{2}(A+B)}\tan.\tfrac{1}{2}c\end{aligned}\right\}\quad\cdots\quad[21].$$

The formulæ [11] and [21] are called "Neper's Analogies," that mathematician having been the first to establish them, and to apply them to the solution of spherical triangles.

(196.) By the first of [10] it follows that $\frac{1}{2}(A+B)$ and $\frac{1}{2}(a+b)$ must be of the same affection. This appears from considering that $\sin.\frac{1}{2}c$ and $\cos.\frac{1}{2}c$ must be both positive, $\because$ $\cos.\frac{1}{2}(A+B)$ and $\cos.\frac{1}{2}(a+b)$ must have the same sign.

The second of [9] proves that $(A-B)$ and $(a-b)$ must be of the same affection, that is, that the greater angle is opposed to the greater side. This has already been proved in (145.).

These considerations are useful in rendering some of the equivocal cases in the solution of triangles determinate.

(197.) There are some formulæ expressing relations be-

tween the sides and angles of a spherical triangle, which, although they be not absolutely requisite in practical problems on the solution of triangles, yet merit attention, were it only for the symmetry and beauty which they exhibit, and the ease with which they may be derived one from another by the analytical process. We shall subjoin, therefore, a few of these formulæ, with the process by which they are established, it being understood that the student, whose only object is the solution of the common cases of spherical triangles, can dispense with the remainder of the present section.

Let

$$n^2 = \sin.s \sin.(s - a)\sin.(s - b)\sin.(s - c),$$
$$\text{N}^2 = -\cos.\text{S} \cos.(\text{S} - \text{A})\cos.(\text{S} - \text{B})\cos.(\text{S} - \text{C}).$$

By multiplying the values found in (182) for the sine, cosine, and tangent of half the angle of a spherical triangle applied to each of the three angles, we obtain

$$\left.\begin{aligned}
\sin.\tfrac{1}{2}\text{A} \sin.\tfrac{1}{2}\text{B} \sin.\tfrac{1}{2}\text{C} &= \frac{\sin.(s-a)\sin.(s-b)\sin.(s-c)}{\sin.a \sin.b \sin.c} \\
&= \frac{n^2}{\sin.s \sin.a \sin.b \sin.c} \\
\cos.\tfrac{1}{2}\text{A} \cos.\tfrac{1}{2}\text{B} \cos.\tfrac{1}{2}\text{C} &= \frac{[\sin.s \sin.s \sin.s \sin.(s-a)\sin.(s-b)\sin.(s-c)]^{\frac{1}{2}}}{\sin.a \sin.b \sin.c} \\
&= \frac{n \sin.s}{\sin.a \sin.b \sin.c} \\
\tan.\tfrac{1}{2}\text{A} \tan.\tfrac{1}{2}\text{B} \tan.\tfrac{1}{2}\text{C} &= \left[\frac{\sin.(s-a)\sin.(s-b)\sin.(s-c)}{\sin.s \sin.s \sin.s}\right]^{\frac{1}{2}} \\
&= \frac{n}{\sin.^2 s}.
\end{aligned}\right\} [22].$$

(198.) The same process applied to the formulæ [13] gives

$$\left.\begin{aligned}
\cos.\tfrac{1}{2}a\cos.\tfrac{1}{2}b\cos.\tfrac{1}{2}c &= \frac{\cos.(s-A)\cos.(s-B)\cos.(s-C)}{\sin.A\sin.B\sin.C}\\
&= \frac{N^2}{-\cos.S\sin.A\sin.B\sin.C}\\
\sin.\tfrac{1}{2}a\sin.\tfrac{1}{2}b\sin.\tfrac{1}{2}c &=\\
\frac{[-\cos.S\cos.S\cos.S\cos.(S-A)\cos.(S-B)\cos.(S-C)]^{\frac{1}{2}}}{\sin.A\sin.B\sin.C}\\
&= \frac{N\cos.S}{\sin.A\sin.B\sin.C}\\
\cot.\tfrac{1}{2}a\cot.\tfrac{1}{2}b\cot.\tfrac{1}{2}c &= \left[-\frac{\cos.(S-A)\cos.(S-B)\cos.(S-C)}{\cos.S\cos.S\cos.S}\right]^{\frac{1}{2}}\\
&= \frac{N}{\cos.^2 S}.
\end{aligned}\right\}[23].$$

(199.) By multiplying [3] by [14], we obtain

$$4Nn = \sin.a\sin.b\sin.c\sin.A\sin.B\sin.C.$$

Also, by dividing the third of [2] by the third of [13],

$$\frac{\sin.A\sin.b\sin.c}{\sin.a\sin.B\sin.C} = \frac{n}{N}.$$

Hence by [4] we have

$$\frac{\sin.A}{\sin.a} = \frac{\sin.B}{\sin.b} = \frac{\sin.C}{\sin.c} = \frac{N}{n},$$

$$\therefore \frac{\sin.a}{\sin.A} = \frac{n}{N}.$$

By comparing this with [3],

$$\frac{N}{n} = \frac{2n}{\sin.a\sin.b\sin.c} \quad \dots \quad [24];$$

and by [14],

$$\frac{n}{N} = \frac{2N}{\sin.A\sin.B\sin.C} \quad \dots \quad [25].$$

Hence we find

$$\left.\begin{aligned}
n &= \tfrac{1}{2}(\sin.^2a\sin.^2b\sin.^2c\sin.A\sin.B\sin.C)^{\frac{1}{3}}\\
N &= \tfrac{1}{2}(\sin.a\sin.b\sin.c\sin.^2A\sin.^2B\sin.^2C)^{\frac{1}{3}}
\end{aligned}\right\} \quad \dots \quad [26].$$

(200.) Symmetrical formulæ for the sines, cosines, and

tangents of s and S are easily obtained. The student will find no difficulty in establishing those of 65 to 70 inclusive in Table VIII.

We shall here investigate the succeeding formulæ of the same table.

By the equations

$$\begin{aligned} 2\sin.s\sin.(s-a) &= \cos.a - \cos.(2s-a) \\ &= \cos.a - \cos.(b+c) \\ &= \cos.a - \cos.b\cos.c + \sin.b\sin.c \\ 2\sin.(s-b)\sin.(s-c) &= \cos.(b-c) - \cos.a \\ &= \cos.b\cos.c + \sin.b\sin.c - \cos.a. \end{aligned}$$

Hence we obtain

$$4n^2 = 1 - \cos.^2a - \cos.^2c - \cos.^2c + 2\cos.a\cos.b\cos.c;$$

and similarly,

$$4N^2 = 1 - \cos.^2A - \cos.^2B - \cos.^2C - 2\cos.A\cos.B\cos.C.$$

These reduced to the half angles, give 71 to 74 inclusive of Table VIII.

The formulæ, to 78 inclusive, follow from the preceding.

(201.) To determine the sines, cosines, and tangents of $s-a$ and $S-A$, we have by the second of [2] and second of [13],

$$\sin.(s-a) = \frac{\sin.b\sin.c\cos.^2\tfrac{1}{2}A}{\sin.s},$$

$$\cos.(S-A) = -\frac{\sin.B\sin.C\sin.^2\tfrac{1}{2}a}{\cos.S},$$

which, by 65 and 68, Table VIII. become, after reduction,

$$\left.\begin{aligned} \sin.(s-a) &= \frac{N}{2\sin.\tfrac{1}{2}A\cos.\tfrac{1}{2}B\cos.\tfrac{1}{2}C} \\ \cos.(S-A) &= \frac{n}{2\cos.\tfrac{1}{2}a\sin.\tfrac{1}{2}b\sin.\tfrac{1}{2}c} \end{aligned}\right\} \cdots [34].$$

By these, combined with the previous results, the remaining formulæ of Table VIII. will easily be established.

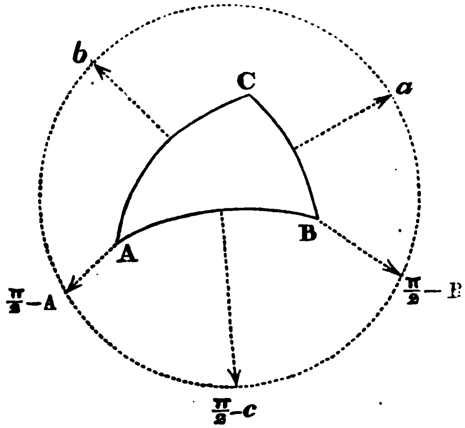
SECTION V.

On the solution of right angled spherical triangles. Neper's rules.

(202.) In a right angled spherical triangle there are five quantities which may become the objects of computation, scil. two angles and the three sides. Any two of these five quantities being known or discoverable, the other three may, in general, be computed. The solution of right angled triangles therefore is resolved into as many cases as there are different combinations of two to be made from five, which are $\frac{5.4}{1.2} = 10$.

To retain the necessary formulæ for these ten cases in the memory would be attended with some difficulty. Neper has, however, by a very ingenious contrivance reduced the ten cases to two, and these so striking and simple, that, when once understood, they will not easily be forgotten. We shall first explain these two celebrated *rules*, and then show that they comprise all the cases.

Let ABC be a spherical triangle right angled at C, and let the sides and angles be expressed as hitherto. Let the triangle be imagined to be placed within a circle, which is merely used to mark the order of certain quantities, to which and to their order thus determined, we shall have occasion to refer. Opposite to the arrows which point from the sides to the surrounding circle are placed a, b, and $\frac{\pi}{2}-c$, that is, the sides and the complement of the hypothenuse. Op-



posite to the arrows which point from the angles are $\frac{\pi}{2} - A$, and $\frac{\pi}{2} - B$, that is, the complements of the angles. These quantities which thus surround the circle are called *circular parts*. Any one of these being taken as *middle part* (M), those which are next to it on each side going round the circle are called *adjacent extremes* (A, A′); and the remaining two are called *opposite extremes* (O, O′). Thus, if $\frac{\pi}{2} - A$ be the middle part, the adjacent extremes will be $\frac{\pi}{2} - c$ and b; and the opposite extremes $\frac{\pi}{2} - B$ and a.

We shall now prove that the two following formulæ are true, and include all the ten cases before mentioned:

$$\sin.M = \tan.A \tan.A',$$
$$\sin.M = \cos.O \cos.O'.$$

These are called *Neper's rules,* and are generally announced thus:

1. "The rectangle under the radius and the sine of the

middle part is equal to the rectangle under the tangents of the adjacent extremes."

2. "The rectangle under the radius and the sine of the middle part is equal to the rectangle under the cosines of the opposite extremes."

The radius being unity, does not appear in the formulæ.

Taking each of the five circular parts as middle successively, and making the proper substitutions in the above formulæ, we obtain the ten following equations; which solve the ten cases of right-angled triangles, and are adapted to logarithmic computation.

1. $\cos.c = \cot.\mathrm{A} \cot.\mathrm{B}$.
2. $\cos.c = \cos.a \cos.b$.
3. $\sin.a = \sin.c \sin.\mathrm{A}$.
4. $\sin.b = \sin.c \sin.\mathrm{B}$.
5. $\cos.\mathrm{A} = \cos.a \sin.\mathrm{B}$.
6. $\cos.\mathrm{B} = \cos.b \sin.\mathrm{A}$.
7. $\cos.\mathrm{A} = \tan.b \cot.c$.
8. $\cos.\mathrm{B} = \tan.a \cot.c$.
9. $\sin.a = \tan.b \cot.\mathrm{B}$.
10. $\sin.b = \tan.a \cot.\mathrm{A}$.

1. In the third formula of [12], Sect. IV., let $\mathrm{C}=\frac{\pi}{2}$, $\because$

$$\cos.\mathrm{A} \cos.\mathrm{B} - \sin.\mathrm{A} \sin.\mathrm{B} \cos.c = 0,$$

$$\therefore \cos.c = \cot.\mathrm{A} \cot.\mathrm{B}.$$

2. In the third formula of [1], Sect. IV. let $\mathrm{C}=\frac{\pi}{2}$, $\because$

$$\cos.c = \cos.a \cos.b.$$

3. and 4. In the equations [4], Sect. IV.,

$$\sin.a \sin.\mathrm{C} = \sin.c \sin.\mathrm{A},$$
$$\sin.b \sin.\mathrm{C} = \sin.c \sin.\mathrm{B},$$

let $\mathrm{C} = \frac{\pi}{2}$, $\because$

$$\sin.a = \sin.c \sin.\mathrm{A}$$
$$\sin.b = \sin.c \sin.\mathrm{B}.$$

5. and 6. In the first and second formulæ of [12], Sect. IV., let $c = \frac{\pi}{2}$, ∵

$$\cos.A = \cos.a \sin.B,$$
$$\cos.B = \cos.b \sin.A.$$

7. and 8. By the second of [15], Sect. IV., when $c = \frac{\pi}{2}$, we have

$$\because \cos.A = \tan.b \cot.c,$$

from which, by changing A into B and b into a, we find

$$\cos.B = \tan.a \cot.c.$$

9. and 10. In the third of [5], Sect. IV., let $c = \frac{\pi}{2}$, and we have

$$\sin.b = \tan.a \cot.A;$$

and by changing A into B and b into a, we find

$$\sin.a = \tan.b \cot.B.$$
$$\because \sin.b = \tan.a \cot.A.$$

(203.) We have thus established *Neper's rules* by proving separately all the several cases which they include. There is no independent or general demonstration of these remarkable theorems, nor is it easy to conceive the process of mind by which their illustrious inventor arrived at them. Professor Woodhouse justly observes, that there are not perhaps in the whole compass of mathematical science rules which more completely attain that which is the proper object of rules, namely, brevity and facility of computation. He might have added, that few, or perhaps no theorems equally general, make such an immediate and permanent impression on the memory.

(204.) Neper's rules, with some modification, may be applied to quadrantal triangles, since such triangles have polar right-angled triangles (138.).

Let $c = \frac{\pi}{2}$. Then one of the polar triangles will have

the angles a, b, $\frac{\pi}{2}$, and the sides A, B, π — C. The circular parts will therefore be A, B, $-\left(\frac{\pi}{2} - \text{C}\right)$, $\frac{\pi}{2} - a$, and $\frac{\pi}{2} - b$. If then in a quadrantal triangle the negative complement of the angle opposed to the quadrantal side, the other two angles and the complements of the sides opposed to them be taken as circular parts surrounding the triangle, Neper's rules will apply.

(205.) By the equation

$$\cos.\text{A} = \cos.a \sin.\text{B},$$

it follows that in a right-angled spherical triangle each of the sides and the angle opposed to it have the same affection. For since B must be less than π, $\therefore \sin.\text{B} > 0$, $\therefore \cos.\text{A}$ and $\cos.a$ must have the same sign, and therefore A and a must have the same affection.

Also, since $\sin.\text{B} < 1 \therefore \cos.\text{A} < \cos.a$. Hence it follows that in a right-angled triangle an oblique angle cannot be less if it be acute, or greater if obtuse, than the opposite side.

It follows from

$$\cos.c = \cos.a \cos.b,$$

that $\cos.a$ and $\cos.b$ have the same or different signs according as c is acute or obtuse, and therefore the sides have the same or different affections according as the hypothenuse is less or greater than 90°.

(206.) By the equation

$$\sin.a = \sin.c \sin.\text{A},$$

it follows that $\sin.a < \sin.c$. Hence, if $a < 90^\circ$, $\therefore a < c$, and if $a > 90^\circ$, $\therefore a > c$. That is, either side of a right-angled spherical triangle is less or greater than the hypothenuse, according as it is less or greater than a quadrant.

(207.) By adding and subtracting the equations

$$\sin.b \cos.a = \sin.c \cos.\text{A},$$
$$\because \sin.a \cos.b = \sin.c \cos.\text{B},$$

obtained from 4 and 5 of (202.), we obtain

$$\sin.(a+b) = 2\sin.c \cos.\tfrac{1}{2}(\text{A}+\text{B})\cos.\tfrac{1}{2}(\text{A}-\text{B}),$$
$$\sin.(a-b) = 2\sin.c \sin.\tfrac{1}{2}(\text{A}+\text{B})\sin.\tfrac{1}{2}(\text{A}-\text{B}),$$
$$\because \frac{\sin.(a-b)}{\sin.(a+b)} = \tan.\tfrac{1}{2}(\text{A}+\text{B})\tan.\tfrac{1}{2}(\text{A}-\text{B}).$$

Hence, since $\sin.(a-b)$ and $\tan.\frac{1}{2}(\text{A}-\text{B})$ must always have the same sign, it follows that $\sin.(a+b)$ and $\tan.\frac{1}{2}(\text{A}+\text{B})$ must also have the same sign, and therefore $a+b$ is $>$ or $<\pi$ according as $\text{A}+\text{B}$ is $>$ or $<\pi$.

(208.) Since the sum of the three angles of every spherical triangle must exceed two right angles, it follows that the sum of the two oblique angles of a right-angled spherical triangle must always exceed one right angle. The difference of the two oblique angles has obviously no minor limit; let it be required to determine its major limit. The two angles A, B, are the sides of a polar triangle, whose remaining side is $\pi - \text{C}$, *i. e.* $\pi - \frac{\pi}{2} = \frac{\pi}{2}$. Since then the difference of two sides of a triangle must be always less than the third side, it is evident that the difference of the two angles A, B, must always be less than 90°. Thus a right angle is the minor limit of the sum, and the major limit of the difference of the oblique angles of a right-angled spherical triangle.

(209.) To apply the formulæ and rules just established to the solution of the cases of right-angled spherical triangles, we shall take each combination of two data, and determine in each case the other three quantities.

1°. Let the hypothenuse c and the angle B be given,

$$\sin.b = \sin.c \sin.\text{B},$$
$$\tan.a = \tan.c \cos.\text{B},$$
$$\cot.\text{A} = \cos.c \tan.\text{B}.$$

2°. Let the hypothenuse c and the angle A be given,

$$\sin.a = \sin.c \sin.\text{A},$$
$$\tan.b = \tan.c \cos.\text{A},$$
$$\cot.\text{B} = \cos.c \tan.\text{A}.$$

3°. Let the hypothenuse c and the side b be given,

$$\sin.\text{B} = \frac{\sin.b}{\sin.c},$$
$$\cos.a = \frac{\cos.c}{\cos.b},$$
$$\cos.\text{A} = \tan.b \cot.c.$$

In this case, sin.B determined by the first formula leaves B equivocal, being indifferently applicable to two supplemental angles. However the doubt is removed by considering that the oblique angles and the sides opposed to them are of the same affection (205.); therefore B and b must be angles of the same kind.

4°. Let the hypothenuse c and the side a be given,

$$\sin.\text{A} = \frac{\sin.a}{\sin.c},$$
$$\cos.b = \frac{\cos.c}{\cos.a},$$
$$\cos.\text{B} = \tan.a \cot.c,$$

A and a being of the same affection as in the last case.

5°. Let the two sides a and b be given,

$$\cos.c = \cos.a \cos.b,$$
$$\tan.\text{A} = \frac{\tan.a}{\sin.b},$$
$$\tan.\text{B} = \frac{\tan.b}{\sin.a}.$$

6°. Let the side a and the opposite angle A be given,

$$\sin.c = \frac{\sin.a}{\sin.\text{A}},$$
$$\sin.b = \tan.a \cot.\text{A},$$
$$\sin.\text{B} = \frac{\cos.\text{A}}{\cos.a}.$$

These values of the sines leave the three unknown quantities doubtful. The uncertainty in some cases may be removed by the considerations in (205.) et seq. The problem is impossible if a and A be of different species (205.).

7°. Let the side b and the opposite angle B be given,

$$\sin.c = \frac{\sin.b}{\sin.B},$$

$$\sin.a = \tan.b \cot.B,$$

$$\sin.A = \frac{\cos.B}{\cos.b},$$

which also leave the sought quantities doubtful, and the solution is impossible when b and B are of different species.

8°. Let the side a and adjacent angle B be given,

$$\tan.c = \frac{\tan.a}{\cos.B},$$

$$\tan.b = \sin.a \tan.B,$$

$$\cos.A = \sin.B \cos.a.$$

9°. Let the side b and the adjacent angle A be given,

$$\tan.c = \frac{\tan.b}{\cos.A},$$

$$\tan.a = \sin.b \tan.A,$$

$$\cos.B = \sin.A \cos.b.$$

10°. Let the two angles A and B given,

$$\cos.c = \cot.A \cot.B,$$

$$\cos.a = \frac{\cos.A}{\sin.B},$$

$$\cos b = \frac{\cos.B}{\sin.A}.$$

(210.) It is in certain cases found expedient to modify the general rules and formulæ for the solution of right-angled triangles which we have here established. It is to be considered that in trigonometrical computations the sought angle is seldom exactly given by the table. It is most generally between the values of two successive terms

of the tables, and can therefore be only computed approximately. The computation should then be so ordered, that the error produced in the result by this circumstance should be as small as possible.

When an arc which is very small is to be determined from its cosine, a small error in the cosine may produce a considerable one in the arc. In such a case the angle should be determined by computing the tangent of its half by the formula

$$\tan.^2\tfrac{1}{2}\varphi = \frac{1-\cos.\varphi}{1+\cos.\varphi}.$$

Thus, if the hypothenuse of a right-angled triangle be sought, the oblique angles being given, we have

$$\cos.c = \cot.\text{A} \cot.\text{B},$$

$$\therefore \tan.^2\tfrac{1}{2}c = \frac{1-\cot.\text{A}\cot.\text{B}}{1+\cot.\text{A}\cot.\text{B}},$$

$$\therefore \tan.^2\tfrac{1}{2}c = \frac{-\cos.(\text{A}+\text{B})}{\cos.(\text{A}-\text{B})}.$$

If the two oblique angles be given, to determine the side opposed to one of them, we have

$$\cos.a \sin.\text{B} = \cos.\text{A},$$

$$\therefore \cos.a = \frac{\cos.\text{A}}{\sin.\text{B}},$$

$$\therefore \tan.^2\tfrac{1}{2}a = \frac{1-\dfrac{\cos.\text{A}}{\sin.\text{B}}}{1+\dfrac{\cos.\text{A}}{\sin.\text{B}}} = \frac{\sin.\text{B}-\cos.\text{A}}{\sin.\text{B}+\cos.\text{A}}.$$

But by (53.) [21] we have

$$\frac{\sin.\text{B}-\sin.\left(\dfrac{\pi}{2}-\text{A}\right)}{\sin.\text{B}+\sin.\left(\dfrac{\pi}{2}-\text{A}\right)} = \frac{\tan.\tfrac{1}{2}(\text{B}-\dfrac{\pi}{2}+\text{A})}{\tan.\tfrac{1}{2}(\text{B}+\dfrac{\pi}{2}-\text{A})},$$

$$\therefore \tan.^2\tfrac{1}{2}a = \tan.[\tfrac{1}{2}(\text{A}+\text{B})-45^{\circ}]\cot.[\tfrac{1}{2}(\text{B}-\text{A})+45^{\circ}].$$

In a similar manner, from

$$\cos.a = \frac{\cos.c}{\cos.b},$$

we may infer

$$\tan.^2\tfrac{1}{2}a = \tan.\tfrac{1}{2}(c - b)\tan.\tfrac{1}{2}(c + b).$$

(211.) If it were required to determine B from the hypothenuse c and the adjacent side a, in

$$\tan.^2\tfrac{1}{2}\text{B} = \frac{1-\cos.\text{B}}{1+\cos.\text{B}};$$

let cos.B be eliminated by the third of 4° (209.), and we find

$$\tan.^2\tfrac{1}{2}\text{B} = \frac{1-\cot.c\tan.a}{1+\cot.c\tan.a} = \frac{\sin.c\cos.a-\sin.a\cos.c}{\sin.c\cos.a+\sin.a\cos.c},$$

$$\therefore \tan.\tfrac{1}{2}\text{B} = \sqrt{\frac{\sin.(c-a)}{\sin.(c+a)}}.$$

Since $\tfrac{1}{2}\text{B} < \frac{\pi}{2}$, the sign + must be given to the radical.

This formula gives all the requisite accuracy when B is very small.

(212.) If the angle which is sought be nearly equal to a right angle, a similar inaccuracy attends the result, when it is derived from its sine, since, in that case, a very small variation in the sine produces a very sensible one in the angle. In this case it is necessary to derive a formula for the tangent from that for the sine.

For example, we have

$$\sin.a = \sin.c\sin.\text{A},$$

$$\tan.^2(45^\circ - \tfrac{1}{2}a) = \frac{1-\sin.a}{1+\sin.a},$$

$$\therefore \tan.^2(45^\circ - \tfrac{1}{2}a) = \frac{1-\sin.c\sin.\text{A}}{1+\sin.c\sin.\text{A}}.$$

Let $\sin.c\sin.\text{A} = \tan.x$, $\therefore$

$$\tan.^2(45^\circ - \tfrac{1}{2}a) = \frac{1-\tan.x}{1+\tan.x} = \tan.(45^\circ - x),$$

$$\because \tan.(45^0 - \tfrac{1}{2}a) = \sqrt{\tan.(45^0 - x)},$$

whence a may be obtained.

When the sought angle is nearly equal to 90°, it is usual to compute, not the angle itself, but the difference between it and 90°. Instances of this will be found in the section on geodetical operations.

(213.) If two right-angled triangles have a common angle or side, a relation will subsist between the angles and sides not common, which will be immediately discoverable by Neper's rules. If the common part be taken as middle part, the product of the tangents or cosines of the adjacent or opposite extremes in the one will be equal to the product of the analogous quantities in the other. Also, if the common part be an extreme, the quote of the sine of the middle part divided by the tangent or cosine of the adjacent or opposite extreme in the one will be equal to the quote of the analogous quantities in the other.

By these principles there is no difficulty in obtaining the following results, which are of considerable use in the solution of problems.

Let the sides and angles of one of the triangles be a, b, c, A, B, and of the other a', b', c', A$'$, B$'$.

I. If $a = a'$,

$$\frac{\tan.b}{\tan.\mathrm{B}} = \frac{\tan.b'}{\tan.\mathrm{B}'}, \quad \sin.\mathrm{A}\sin.c = \sin.\mathrm{A}'\sin.c',$$

$$\frac{\sin.\mathrm{B}}{\cos.\mathrm{A}} = \frac{\sin.\mathrm{B}'}{\cos.\mathrm{A}'}, \quad \cos.\mathrm{B}\tan.c = \cos.\mathrm{B}'\tan.c',$$

$$\frac{\cos.c}{\cos.b} = \frac{\cos.c'}{\cos.b'}, \quad \sin.b\tan.\mathrm{A} = \sin.b'\tan.\mathrm{A}'.$$

II. If A = A$'$,

$$\frac{\sin.c}{\sin.a} = \frac{\sin.c'}{\sin.a'}, \quad \sin.\mathrm{B}\cos.a = \sin.\mathrm{B}'\cos.a',$$

$$\frac{\cos.\mathrm{B}}{\cos.b} = \frac{\cos.\mathrm{B}'}{\cos.b'}, \quad \cos.c\tan.\mathrm{B} = \cos.c'\tan.\mathrm{B}',$$

$$\frac{\sin.b}{\tan.a}=\frac{\sin.b'}{\tan.a'}, \quad \frac{\tan.c}{\tan.b}=\frac{\tan.c'}{\tan.b'}$$

III. If $c = c'$,

$$\frac{\sin.a}{\sin.A}=\frac{\sin.a'}{\sin.A'}, \quad \cos.a\cos.b = \cos.a'\cos.b',$$

$$\frac{\tan.a}{\cos.A}=\frac{\tan.a'}{\cos.A'}, \quad \tan.A\tan.B = \tan.A'\tan.B'.$$

SECTION VI.

Examples on right-angled spherical triangles.

PROP. LXVII.

(214.) *Given the hypothenuse of a right-angled spherical triangle, and the sum or difference of the sides, to compute the sides themselves.*

By Neper's rules,

$$\cos.c = \cos.a\cos.b,$$

$$\because 2\cos.c = \cos.(a-b) + \cos.(a+b);$$

when the sum of the sides is given, this equation will determine the difference, and *vice versâ;* and the sum and difference being known, the sides themselves may be found.

PROP. LXVIII.

(215.) *Given one side and the sum or difference of the hypothenuse and the other side, to find the hypothenuse, the side and the angle opposed to it.*

By (53.),

$$\tan.\tfrac{1}{2}(c-b)\tan.\tfrac{1}{2}(c+b)=\frac{\cos.b-\cos.c}{\cos.b+\cos.c},$$

$$\cos.c = \cos.a\cos.b,$$

$$\therefore \tan.\tfrac{1}{2}(c-b)\tan.\tfrac{1}{2}(c+b) = \frac{\cos.b - \cos.a\cos.b}{\cos.b + \cos.a\cos.b}$$

$$= \frac{1-\cos.a}{1+\cos.a},$$

$$\therefore \tan.\tfrac{1}{2}(c-b)\tan.\tfrac{1}{2}(c+b) = \tan.^2\tfrac{1}{2}a;$$

when either $c - b$ or $c + b$ is given, this equation will determine the other.

To determine the angle B, we have

$$\sin.c\sin.B = \sin.b,$$

$$\tan.^2(45^0 + \tfrac{1}{2}B) = \frac{1+\sin.B}{1-\sin.B}$$

$$= \frac{\sin.c + \sin.b}{\sin.c - \sin.b}$$

$$= \frac{\tan.\frac{1}{2}(c+b)}{\tan.\frac{1}{2}(c-b)}.$$

By multiplying and dividing this by the equation found in the last case, we obtain

$$\tan.\tfrac{1}{2}a\tan.(45^0 + \tfrac{1}{2}B) = \tan.\tfrac{1}{2}(c+b)$$

$$\tan.\tfrac{1}{2}a\cot.(45^0 + \tfrac{1}{2}B) = \tan.\tfrac{1}{2}(c-b).$$

The former gives the value of B when $c + b$ is given, and the latter when $c - b$ is given.

PROP. LXIX.

(216.) *Given the sum or difference of the hypothenuse and one side, and the included angle, to find the hypothenuse and side severally.*

To determine b and c separately, we have

$$\tan.^2\tfrac{1}{2}A = \frac{1-\cos.A}{1+\cos.A},$$

$$\cos.A = \cot.c\tan.b = \frac{\tan.b}{\tan.c},$$

$$\therefore \tan.^2\tfrac{1}{2}A = \frac{\tan.c - \tan.b}{\tan.c + \tan.b} = \frac{\sin.(c-b)}{\sin.(c+b)},$$

$$\therefore \sin.(c+b)\tan.^2\tfrac{1}{2}A = \sin.(c-b).$$

This equation determines either $(c + b)$ or $(c - b)$ when the other is given. Hence c and b may be found severally.

PROP. LXX.

(217.) *Given an angle* (A) *and the sum or difference of the opposite side and the hypothenuse, to find the sides of the triangle severally.*

By the results of (215.),

$$\tan.\tfrac{1}{2}(c - a) = \cot.^2(45^0 + \tfrac{1}{2}A)\tan.\tfrac{1}{2}(c + a),$$
$$\tan.\tfrac{1}{2}b = \cot.(45^0 + \tfrac{1}{2}A)\tan.\tfrac{1}{2}(c + a),$$
$$\tan.\tfrac{1}{2}b = \tan.(45^0 + \tfrac{1}{2}A)\tan.\tfrac{1}{2}(c - a),$$

which equations solve the problem.

PROP. LXXI.

(218.) *Given the hypothenuse and the sum or difference of the angles, to find the angles.*

By (46.),

$$\frac{\cos.(A-B)}{\cos.(A+B)} = \frac{\cot.A\cot.B+1}{\cot.A\cot.B-1},$$

$$\cot.A\cot.B = \cos.c,$$

$$\therefore \frac{\cos.(A-B)}{\cos.(A+B)} = \frac{\cos.c+1}{\cos.c-1} = -\cot.^2\tfrac{1}{2}c,$$

which equation solves the problem.

PROP. LXXII.

(219.) *Given the sun's longitude* (c) *and the obliquity of the ecliptic* (A), *to find his right ascension* (b) *and declination* (a).

By the last section,

$$\tan.b = \cos.\text{A} \tan.c,$$
$$\sin.a = \sin.\text{A} \sin.c.$$

In this case a is not equivocal, since it cannot exceed $\text{A} = 23^\circ\ 28'$.

PROP. LXXIII.

(220.) *Given the sun's declination* (D), *to find the time of sunrise in a given latitude.*

Let the latitude be L. There is a right-angled triangle formed by the sun's polar distance $\left(\frac{\pi}{2} - \text{D}\right)$ at sunrise and the altitude of the pole L. Let H be the hour angle sought,

$$\cos.\text{H} = \tan.\text{L} \cot.\left(\frac{\pi}{2} - \text{D}\right) = \tan.\text{L} \tan.\text{D}.$$

SECTION VII.

Solution of oblique angled spherical triangles.

(221.) In spherical triangles there are six quantities, any three of which being given, the other three may, in general, be computed, the three angles being sufficient data, which is not the case in plane triangles. There would then be as many distinct systems of data in the solution of oblique spherical triangles as there could be combinations of three made from six $= \frac{6.5.4}{1.2.3} = 20$. They may be, however, reduced to a smaller number of more comprehensive classes. It is obvious that the three data must always come under some one of the following systems:

1°. The three sides.
2°. The three angles.
3°. Two sides and an angle.
4°. Two angles and a side.

The last two cases each include two. In the former, the given angle may either be placed between the given sides, or opposite to one of them; and in the latter, the given side may either be placed between the given angles, or opposite to one of them. Thus all possible systems of data for oblique triangles are reduced to the following six:

I. The three sides.
II. The three angles.
III. Two sides and the included angle.
IV. Two angles and the included side.
V. Two sides and the angle opposite one of them.
VI. Two angles and the side opposite one of them.

We shall consider these cases successively.

I.

Given the three sides.

(222.) The values of the angles may be determined by four distinct formulæ deduced from the results of (182.) as follow:

1°. $\sin.\frac{1}{2}A = \sqrt{\frac{\sin.(s-b)\sin.(s-c)}{\sin.b\sin.c}}$,

2°. $\cos.\frac{1}{2}A = \sqrt{\frac{\sin.s\sin.(s-a)}{\sin.b\sin.c}}$.

3°. $\sin.A = \frac{2}{\sin.b\sin.c}\sqrt{\sin.s\sin.(s-a)\sin.(s-b)\sin.(s-c)}$.

4°. $\tan.\frac{1}{2}A = \sqrt{\frac{\sin.(s-b)\sin.(s-c)}{\sin.s\sin.(s-a)}}$.

These formulæ are all suited to logarithmic calculation.

It would appear that in cases where the three angles are

required, the third would be the most convenient, as the same radical would occur in the values of the sines of all the angles. This, however, is not found so in practice. The last formula is the most generally useful, being applicable to angles of all magnitudes. The second formula does not give sufficient accuracy when the angle A is small, and the first is subject to the same objection when $\frac{1}{2}$A is nearly $= 90°$, or when A is nearly $= 180°$. As it seldom happens that the angle is nearly 180°, the first formula is most frequently used.

(223.) 5°. The angle might also be obtained from

$$\cos.A = \frac{\cos.a - \cos.b\cos.c}{\sin.b\sin.c}$$

by means of a subsidiary angle. Let

$$\cos.\theta = \cos.b\cos.c.$$

It is always possible to assign a value to θ, which will fulfil this condition, since $\cos.b\cos.c < 1$. Also, it may be observed, that θ is acute or obtuse, according as b and c are of the same or different affections. Hence

$$\cos.A = \frac{\cos.a - \cos.\theta}{\sin.b\sin.c} = \frac{2\sin.\frac{1}{2}(\theta - a)\sin.\frac{1}{2}(\theta + a)}{\sin.b\sin.c},$$

in which A is $>$ or $< \frac{\pi}{2}$, according as θ is $>$ or $< a$.

The value of θ is the hypothenuse of a right-angled spherical triangle, whose sides are b and c.

(224.) 6°. This problem may also be solved by supposing

$$\tan.\theta = \frac{\cos.b\cos.c}{\sin.a},$$

and by putting the formula

$$\cos.A = \frac{\cos.a - \cos.b\cos.c}{\sin.b\sin.c}$$

under the form

$$\cos.\mathrm{A} = \frac{\cos.a - \dfrac{\cos.b\cos.c}{\sin.a}.\sin.a}{\sin.b\sin.c}.$$

Hence we obtain

$$\cos.\mathrm{A} = \frac{\cos.a - \tan.\theta\sin.a}{\sin.b\sin.c}$$

$$\therefore \cos.\mathrm{A} = \frac{\cos.a\cos.\theta - \sin.a\sin.\theta}{\sin.b\sin.c\cos.\theta} = \frac{\cos.(a+\theta)}{\sin.b\sin.c\cos.\theta},$$

which is suited to logarithms.

(225.) 7°. The angle A may be determined by the segments x, y, into which the side a opposed to it is divided by the perpendicular from the vertex of the angle A.

By (213.) we have

$$\frac{\cos.x}{\cos.y} = \frac{\cos.b}{\cos.c},$$

the perpendicular being a common side of the two right-angled triangles. Hence

$$\frac{\cos.x - \cos.y}{\cos.x + \cos.y} = \frac{\cos.b - \cos.c}{\cos.b + \cos.c},$$

$$\therefore \tan.\tfrac{1}{2}(y-x)\tan.\tfrac{1}{2}(y+x) = \tan.\tfrac{1}{2}(c-b)\tan.\tfrac{1}{2}(c+b),$$

$$\therefore \tan.\tfrac{1}{2}(y-x) = \tan.\tfrac{1}{2}(c-b)\tan.\tfrac{1}{2}(c+b)\cot.\tfrac{1}{2}(y+x),$$

$$\tfrac{1}{2}(y \pm x) = a.$$

The half sum and half difference of the segments being given, the segments themselves may be found. The segments of the base being determined, the base angles may be computed by the formulæ,

$$\cos.\mathrm{B} = \cot.c\tan.\tfrac{1}{2}(a + d),$$

$$\cos.\mathrm{C} = \cot.b\tan.\tfrac{1}{2}(a - d),$$

where $d = y - x$.

By attending to the signs of the several quantities, it will not be necessary to distinguish the cases where the perpendicular falls within and without the base a.

II.

Given the three angles *.

(226.) In this case, like the last, the sides may be determined by any one of four formulæ deduced from the results of (195).

$$1^{\circ}.\ \sin.\tfrac{1}{2}a = \sqrt{\frac{-\cos.S\cos.(S-A)}{\sin.B\sin.C}}.$$

$$2^{\circ}.\ \cos.\tfrac{1}{2}a = \sqrt{\frac{\cos.(S-B)\cos.(S-C)}{\sin.B\sin.C}}.$$

$$3^{\circ}.\ \sin.a = \frac{2}{\sin.B\sin.C}\sqrt{-\cos.S\cos.(S-A)\cos(S-B)\cos.(S-C)}.$$

$$4^{\circ}.\ \cot.\tfrac{1}{2}a = \sqrt{-\frac{\cos.(S-B)\cos.(S-C)}{\cos.S\cos.(S-A)}}.$$

The same observations apply here as in the former case.

(227.) This problem may be also solved in a manner analogous to the fifth method used in the first case. We have by [12] (195.),

$$\cos.a = \frac{\cos.A+\cos.B\cos.C}{\sin.B\sin.C}.$$

Let $\cos.\theta = \cos.B\cos.C$, $\because$

$$\cos.a = \frac{\cos.A+\cos.\theta}{\sin.B\sin.C} = \frac{2\cos.\frac{1}{2}(A+\theta)\cos.\frac{1}{2}(A-\theta)}{\sin.B\sin.C}.$$

(228.) The sixth method used in the first case may be applied here in the same manner.

Let

$$\tan.\theta = \frac{\cos.B\cos.C}{\sin.A},$$

---

* Je n'ai jamais trouvé l'application de ce probleme qu'une seule fois, et encore je pouvais m'en passer. DELAMBRE.

$$\therefore \cos.a = \frac{\cos.(\text{A} - \theta)}{\sin.\text{B} \sin.\text{C} \cos.\theta}.$$

Also, if D be the difference of the angles which the perpendicular on a makes with the sides b and c, we have

$$\cos.c = \cot.\text{B} \cot.\tfrac{1}{2}(\text{A} + \text{D}),$$
$$\cos.b = \cot.\text{C} \cot.\tfrac{1}{2}(\text{A} - \text{D}).$$

(229.) The analogy which subsists between the formulæ for plane and spherical triangles appears to be in some degree broken by this case, in which the three angles are sufficient to determine the sides in a spherical triangle, although they only determine the proportion of the sides in a plane triangle. There is, however, no real difference between the cases. The three angles of a spherical triangle do not determine the actual lengths of the sides, but only the number of degrees which the sides contain. This, therefore, determines only the proportion of the sides. The absolute lengths must be deduced from the knowledge either of the absolute length of the radius of the sphere or of some line which has a known relation to the triangle. In the same manner in a plane triangle, if, besides the three angles, the radius of the circumscribed or inscribed circle be given, or any other line having a known relation to the triangle, the sides may be computed. Thus it appears that the analogy is perfect. As much can be determined respecting a plane triangle as of a spherical triangle, when its three angles are given.

III.

Given two sides and the included angle.

(230.) The remaining angles may be deduced from the following formulæ established in (190).

$$\left.\begin{aligned}\tan.\tfrac{1}{2}(A+B) &= \frac{\cos.\frac{1}{2}(a-b)}{\cos.\frac{1}{2}(a+b)}\cot.\tfrac{1}{2}C \\ \tan.\tfrac{1}{2}(A-B) &= \frac{\sin.\frac{1}{2}(a-b)}{\sin.\frac{1}{2}(a+b)}\cot.\tfrac{1}{2}C\end{aligned}\right\}.$$

Having determined the sum and difference of the remaining angles by these formulæ, the angles may be immediately found by addition and subtraction.

(231.) It may happen that one angle only is required, in which case, a more expeditious method may be given.

By the third of [5], Sect. IV., we have

$$\cos.C\cos.b + \sin.C\cot.A - \sin.b\cot.a = 0,$$

$$\because \cot.A = \frac{\sin.b\cot.a}{\sin.C} - \cot.C\cos.b,$$

$$\cot.A = \cot.C\left[\frac{\sin.b\cot.a}{\cos.C} - \cos.b\right].$$

Let

$$\frac{\cot.a}{\cos.C} = \cot.\theta.$$

In which case θ will be the side of a right-angled triangle, of which a is the hypothenuse, and C the included angle.

Hence

$$\cot.A = \cot.C[\sin.b\cot.\theta - \cos.b]$$

$$= \frac{\cot.C\sin.(b-\theta)}{\sin.\theta}.$$

(232.) To determine the remaining side c, having previously determined the angles A, B, as above, we have the formula

$$\sin.c = \frac{\sin.a}{\sin.A}\sin.C.$$

But it is frequently required from the knowledge of two sides and the included angle, to be able to determine only

the remaining side, in which case the process of first determining the angles, and thence the side, would be circuitous and elaborate. It is therefore necessary that we should be furnished with formulæ expressing the value of one side of a triangle in terms of the opposite angle and remaining sides, and that the formulæ should be such as may admit of logarithmic computation. There are several methods of solving this problem.

First method.

(233.) Let the formula

$$\cos.c = \cos.a \cos.b + \sin.a \sin.b \cos.\mathrm{c}$$

be put under the form

$$\cos.c = \cos.a[\cos.b + \tan.a \sin.b \cos.\mathrm{c}],$$

$$\cos.c = \cos.a[\cos.b + \sin.b \cot.\theta],$$

where

$$\cot.\theta = \tan.a \cos.\mathrm{c},$$

in which case θ is the oblique angle of a right-angled triangle, the remaining oblique angle being a, and the hypothenuse c.

Hence

$$\cos.c = \cos.a \frac{\sin.\theta \cos.b + \sin.b \cos.\theta}{\sin.\theta} = \frac{\cos.a \sin.(\theta + b)}{\sin.\theta};$$

or, if

$$\tan.\theta = \tan.a \cos.\mathrm{c},$$

$$\cos.c = \frac{\cos.a \cos.(\theta - b)}{\cos.\theta}.$$

In which case, a and θ would be the hypothenuse and side, and c the included angle of the right-angled triangle.

Second method.

(234.) By (182.) we have

$$\cos.c = \cos.(a-b) - 2\sin.^2\tfrac{1}{2}\mathrm{c} \sin.a \sin.b$$

$$= \cos.(a-b)\left[1 - \frac{2\sin.^2\frac{1}{2}\mathrm{c} \sin.a \sin.b}{\cos.(a-b)}\right] = \cos.(a-b)\mathrm{M}^2,$$

The method of computing the value of a factor of the form M^2, we shall presently explain. For the present, if M^2 be supposed to be known, this formula solves the problem.

Third method.

(235.) By (182.) we have

$$\cos.c = \cos(a+b) + 2\cos.^2\tfrac{1}{2}c \sin.a \sin.b,$$

$$= \cos.(a+b)\left[1 + \frac{2\cos.^2\frac{1}{2}c\sin.a\sin.b}{\cos.(a+b)}\right] = \cos.(a+b)M^2.$$

which also solves the problem when M is known.

Fourth method.

(236.) By Table VIII., 49, we have

$$\sin.\tfrac{1}{2}c = \sin.\tfrac{1}{2}(a-b)\cos.\tfrac{1}{2}c[1 + \frac{\sin^2\frac{1}{2}(a+b)}{\sin^2\frac{1}{2}(a-b)}\tan^2\tfrac{1}{2}c]^{\frac{1}{2}},$$

$$\therefore \sin.\tfrac{1}{2}c = \sin.\tfrac{1}{2}(a-b)\cos.\tfrac{1}{2}c \,.\, M.$$

Fifth method.

(237.) By Table VIII., 50,

$$\cos.\tfrac{1}{2}c = \cos.\tfrac{1}{2}(a-b)\cos.\tfrac{1}{2}c[1 + \frac{\cos.^2\frac{1}{2}(a+b)}{\cos.^2\frac{1}{2}(a-b)}\tan.^2\tfrac{1}{2}c]^{\frac{1}{2}},$$

$$\cos.\tfrac{1}{2}c = \cos.\tfrac{1}{2}(a-b)\cos.\tfrac{1}{2}c \,.\, M.$$

(238.) The quantities represented by M^2 in these formulæ are of the form

$$M^2 = 1 \pm N.$$

If the sign be $+$, let

$$N = \tan.^2\theta,$$

$$\therefore M^2 = \sec.^2\theta \quad \therefore M = \sec.\theta.$$

The value of M in the fourth and fifth methods may be always thus determined, since the quantities represented by N are in these cases essentially positive.

If the sign be $-$, it will be necessary to consider separately the cases where $N < 1$ and $N > 1$.

1°. If $N < 1$. Let

$$N = \cos.^2\theta,$$
$$\therefore M = \sin.\theta.$$

2°. If $N > 1$. Let

$$N = \sec.^2\theta,$$
$$\therefore M^2 = 1 - \sec.^2\theta,$$
$$\therefore M^2 = -\tan.^2\theta.$$

The nature of the subsidiary angle used in the second and third methods must therefore be determined by the particular circumstances of the case.

Sixth method.

(239.) By (182.) we have

$$\cos.c = \cos.\ (a + b) + 2\sin.a\sin.b\cos.^2\tfrac{1}{2}c,$$
$$\therefore \sin.^2\tfrac{1}{2}c = \sin.^2\tfrac{1}{2}(a + b) - \sin.a\sin.b\cos.^2\tfrac{1}{2}c,$$
$$\sin.^2\tfrac{1}{2}c = \sin.^2\tfrac{1}{2}(a + b) - \sin.^2\theta,$$

where

$$\sin.^2\theta = \sin.a\sin.b\cos.^2\tfrac{1}{2}c.$$

It is always possible to assign to θ a value which will fulfil this equation, since a and b are each $< \pi$, $\therefore \sin.a$ and $\sin.b$ are both positive. Also, $\sin.a$, $\sin.b$, $\cos.^2\frac{1}{2}c$, being all < 1, their product must be < 1.

Hence

$$\sin.^2\tfrac{1}{2}c = \sin.[\tfrac{1}{2}(a + b) + \theta]\sin.[\tfrac{1}{2}(a + b) - \theta]\ *.$$

(240.) This method suggested by Laplace does not give a sufficiently accurate result in practical computation when the side c is nearly 180°. Although this seldom occurs, yet a similar formula suited to this case is easily established.

* See *Laplace, Mec. Cel.* liv. ii. p. 227.

By (182.) we have

$$\cos.c = \cos.(a - b) - 2\sin.a\sin.b\sin.^2\tfrac{1}{2}C,$$
$$\cos.^2\tfrac{1}{2}c = \cos.^2\tfrac{1}{2}(a - b) - \sin.a\sin.b\sin.^2\tfrac{1}{2}C,$$
$$\because \cos.^2\tfrac{1}{2}c = \cos.^2\tfrac{1}{2}(a - b) - \sin.^2\theta,$$

where

$$\sin.^2\theta = \sin.a\sin.b\sin.^2\tfrac{1}{2}C.$$

Hence by (46.)

$$\cos.^2\tfrac{1}{2}c = \cos.[\tfrac{1}{2}(a - b) + \theta]\cos.[\tfrac{1}{2}(a - b) - \theta].$$

IV.

Given two angles and the included side.

(241.) The sides opposed to the given angles may be determined by the following formulæ:

$$\left.\begin{aligned}\tan.\tfrac{1}{2}(a + b) &= \frac{\cos.\tfrac{1}{2}(A-B)}{\cos.\tfrac{1}{2}(A+B)}\tan.\tfrac{1}{2}c\\ \tan.\tfrac{1}{2}(a - b) &= \frac{\sin.\tfrac{1}{2}(A-B)}{\sin.\tfrac{1}{2}(A+B)}\tan.\tfrac{1}{2}c\end{aligned}\right\}$$

The sum and difference of the sides being found, the sides themselves can be determined by addition and subtraction.

(242.) If one side only be required, a formula for the solution may be derived from the result of (231.) by substituting $\pi - A$, $\pi - C$, and $\pi - c$, for a, c, and C.

The result is

$$\cot.a = \cot.c[\sin.B\cot.\theta + \cos.B]$$
$$= \frac{\cot.c\sin.(B + \theta)}{\sin.\theta},$$

where

$$\cot.\theta = \frac{\cot.A}{\cos.c}.$$

(243.) To determine the remaining angle, having previously determined the side a or b as above, we have

$$\sin.C = \frac{\sin.A}{\sin.a}\sin.c,$$

$$\sin.C = \frac{\sin.B}{\sin.b}\sin.c.$$

But for the same reasons, as in the last case, it is necessary that we should establish methods for determining C independently of a and b. We can immediately deduce, by means of the properties of the supplemental polar triangle, methods for determining C analogous to those for determining c in the last case.

First method.

(244.) Let

$$\cot.\theta = \tan.A \cos.c,$$

$$\because (233.),\ \cos.C \doteq \frac{\cos.A \sin.(\theta - B)}{\sin.\theta}:$$

or, if

$$\tan.\theta = \tan.A \cos.c,$$

$$\because \cos.C = \frac{\cos.A \cos.(\theta + B)}{\sin.\theta}.$$

Second method.

(245.) By (234.) we obtain

$$\cos.C = -\cos.(A - B)\left[1 - \frac{2\cos.^2\frac{1}{2}c\sin.A \sin.B}{\cos.(A - B)}\right],$$

or

$$\cos.C = -\cos.(A - B)M^2.$$

Third method.

(246.) By (235.) we have

$$\cos.c = -\cos.(A+B)\left[1+\frac{2\sin.^2\frac{1}{2}c\sin.A\sin.B}{\cos.(A+B)}\right],$$

$$\cos.c = -\cos.(A+B)M^2.$$

Fourth method.

(247.) By (236.) we have

$$\cos.\tfrac{1}{2}c = -\sin.\tfrac{1}{2}(A-B)\sin.\tfrac{1}{2}c\left[1+\frac{\sin.^2\frac{1}{2}(A+B)}{\sin.^2\frac{1}{2}(A-B)}\cot.^2\tfrac{1}{2}c\right]^{\frac{1}{2}},$$

$$\cos.\tfrac{1}{2}c = -\sin.\tfrac{1}{2}(A-B)\sin.\tfrac{1}{2}c.M.$$

Fifth method.

(248.) By (237.),

$$\sin.\tfrac{1}{2}c = \cos.\tfrac{1}{2}(A-B)\sin.\tfrac{1}{2}c\left[1+\frac{\cos.^2\frac{1}{2}(A+B)}{\cos.^2\frac{1}{2}(A-B)}\cot.^2\tfrac{1}{2}c\right]^{\frac{1}{2}},$$

$$\sin.\tfrac{1}{2}c = \cos.\tfrac{1}{2}(A-B)\sin.\tfrac{1}{2}c.M.$$

The quantities represented by M in the last five methods are computed as in (238.)

Sixth method.

(249.) By (239.) we have

$$\cos.^2\tfrac{1}{2}c = \sin.^2\tfrac{1}{2}(A+B) - \sin.^2\theta,$$

$$\sin.^2\theta = \sin.A\sin.B\sin.^2\tfrac{1}{2}c,$$

$$\cos.^2\tfrac{1}{2}c = \sin.[\tfrac{1}{2}(A+B)+\theta]\times\sin.[\tfrac{1}{2}(A+B)-\theta].$$

This formula is not applicable with accuracy when c is small. We may derive one, however, from the formula established in (240.), which gives

$$\sin.^2\theta = \sin.A\sin.B\cos.^2\tfrac{1}{2}c,$$

$$\sin.^2\tfrac{1}{2}c = \cos.[\tfrac{1}{2}(A-B)+\theta]\times\cos.[\tfrac{1}{2}(A-B)-\theta].$$

(250.) It will contribute much to the clearness of the investigation of the fifth and sixth cases, to determine previously under what conditions it is possible that a side of a spherical triangle and the angle opposed to it can be of

different species. For this purpose, let the equation

$$\cos.a - \cos.\mathrm{A} \sin.b \sin.c - \cos.b \cos.c = 0$$

be expressed thus,

$$\cos.a - \cos.b \cos.c = \cos.\mathrm{A} \sin.b \sin.c.$$

If a and A be of different species, $\cos.a$ and $\cos.\mathrm{A}$ have different signs, and since $\sin.b \sin.c$ is necessarily positive, $\cos.a$ and $\cos.\mathrm{A} \sin.b \sin.c$ have different signs. Hence it follows that $\cos.a - \cos.b \cos.c$ must have a sign different from that of $\cos.a$, and therefore $\cos.a < \cos.b \cos.c$. But since $\cos.b$ and $\cos.c$ are each less than unity, their product is less than either of them. Hence $\cos.a$ is less than $\cos.b$ or $\cos.c$, and therefore $\sin.a$ is greater than $\sin.b$ or $\sin.c$.

From this reasoning it appears that no side of a spherical triangle can differ in species from its opposite angle, except that side whose sine is greater than the sines of the other sides. By [4], Sect. IV., it appears that the sine of the angle opposed to such a side is greater than the sines of the remaining angles.

From this it immediately follows that the sides of a right-angled triangle and those of a quadrantal triangle are of the same species as the angles opposed to them, since the sine of the right angle in the one is necessarily greater than the sines of the other angles, and the sine of the quadrantal side in the other is greater than the sines of the other sides.

Also, it follows that the sides of an isosceles triangle must be of the same species as the opposite angles, and that the sides and angles of an equilateral triangle are of the same species.

If $a > 90^\circ$ and $\mathrm{A} < 90^\circ$, $\because \cos.a < 0$, and therefore, $\cos.b \cos.c$ must necessarily be negative, and b and c of different species; therefore, also, B and C are of different species.

Also, if $a < 90^\circ$ and $\mathrm{A} > 90^\circ$, $\because \cos.a > 0$, and there-

fore, $\cos.b \cos.c$ must be positive, and, consequently, b and c and also B and C must be of the same species.

Hence in a spherical triangle, if an acute angle be opposed to an obtuse side, the remaining sides must be of different species, and each of them being of the same species as the angle opposed to it, the remaining angles must also be of different species. And if an obtuse angle be opposed to an acute side, the remaining sides and angles must all be of the same species.

V.

Given two sides and the angle opposed to one of them.

(251.) Let the sides be a, b, and the angle A.

To determine the angle B, we have the equation

$$\sin.B = \frac{\sin.b}{\sin.a}\sin.A = m\sin.A.$$

In order that it should be possible to obtain an angle B whose sine is $= m\sin.A$, it is necessary that $m\sin.A$ should not be a number greater than unity. It may therefore be concluded, that if $m\sin.A > 1$, the problem admits of no solution, and the data are inconsistent with each other.

If $m\sin.A = 1$, $\therefore \sin.B = 1$, $\therefore B = 90°$. It appears, however, that even in this case, if A and a be of different species, the problem admits of no solution. The cause why this circumstance does not appear from the above equation, and why a real value of the unknown quantity is obtained when the problem is impossible, is, that the sine of a is the same as that of its supplement, and as the sines only enter the equation, the problem stated analytically by the equation is more general than the problem proposed geometrically, inasmuch, as the former includes the triangle whose side is $\pi - a$, as well as that whose side is a.

If a and A be of the same species, or not being so, if they be made so by changing either of them into its supplement, the problem is determinate, $B = 90°$, and the values of C and c may be determined as in the solution of that case of right-angled triangles where the hypothenuse and one side are given.

If $m\sin.A < 1$, an angle less than 90^0 may always be found whose sine $= m\sin.A$. Let ω be such an angle, and since

$$\sin.\omega = \sin.(\pi - \omega),$$
$$\because \sin.\omega = m\sin.A,$$
$$\sin.(\pi - \omega) = m\sin.A.$$

Thus it appears that in general there are two values, ω and $(\pi - \omega)$ of B, which both satisfy the equation

$$\sin.B = m\sin.A;$$

but it does not therefore necessarily follow that both these values of B will solve the problem. In the problem, as proposed geometrically, the data are the two sides a, b, and the angle A; and the quantity sought is the angle B. In the problem expressed analytically by the equation

$$\sin.B = \frac{\sin.b}{\sin.a}\sin.A,$$

the data are the *sines* of the sides a, b, and the *sine* of the angle A; and the sought quantity is not as before the *angle* B, but its *sine*. This is much more general than the problem proposed, because the sines of a, b, and A, are also the sines of their supplements, and therefore the problem stated analytically includes the triangles, whose sides are, one or both, the supplements of a and b, and whose angle is the supplement of A. It follows, therefore, that of the two values of B determined by the equation, either or both may be the angle, not of the triangle, whose sides are a, b, and whose angle is A, but whose sides or angle are supplemental to any or all of these. It is true that all these

triangles are found by the same great circles of the sphere (131.); but they are not all triangles having the given sides and angle.

Since then the analytical statement of the problem includes cases which the problem proposed does not, it becomes necessary to determine by means of the data of the problem itself, scil. the sides a, b, and the angle A, whether either or both of the values of B determined analytically will solve the problem, or whether they only solve those cases not contained in the problem itself, although included in the analytical expression of it.

The two values of B, scil. ω and $\pi - \omega$ being supplemental, are of different species. Now it follows from (250.), if $\sin.b < \sin.a$, that is, if $m < 1$, that b and B must be of the same species. Since, then, b is given, it is at once determined, that of the two values of B, that only solves the problem which is of the same species with b. The other value corresponds to the spherical triangle, whose side is $\pi - b$ (131.)

If $\sin.b > \sin.a$, or $m > 1$, but at the same time $m\sin.\text{A} < 1$, both values of B will be admissible, if a and A be of the same species. For in that case, $\sin.b$ being the greater of the two given sines, the angle B may be either of the same or different species as b, and therefore either of the supplemental values ω or $\pi - \omega$ will solve the problem (250.). Since, however, $\sin.a$ is the lesser of the two given sines, it is necessary that A and a should be of the same species (250.), otherwise no triangle could be constructed with the proposed data. In this case, therefore, the problem would be impossible, but might be rendered possible by changing either a or A into its supplement.

It may therefore be always determined immediately from the data, whether the problem be impossible, and if possible, whether it admit of one or of two solutions.

The following tests may be immediately deduced from what has been just established. See Table IX.

1°. If $\frac{\sin.a}{\sin.b} < \sin.A$.	The problem admits of no solution.
2°. If $\frac{\sin.a}{\sin.b} = \sin.A$.	If a and A be of same species, B $= 90°$, and there is but one solution. If a and A be of different species, there is no solution.
3°. If $\frac{\sin.a}{\sin.b} > \sin.A$ and < 1.	If a and A be of the same species, there are two solutions in which the values of B are supplemental. If a and A be of different species, there is no solution.
4°. If $\frac{\sin.a}{\sin.b}$ not < 1.	There is but one solution, B being of the same species with b.

Thus it appears that the circumstances of the problem depend on the relation which the quantity $\frac{\sin.a}{\sin.b}$ bears to sin.A and unity. If its value be less than the former, there will be no solution, and the problem is impossible; if greater than the latter, there will be but one, and the problem is determinate. If its value lie between those limits, there are either two solutions or none, and therefore the problem is either doubtful or impossible, according as a and A are of the same or different species.

(252.) In determining the other parts C and c of the triangle, it will be unnecessary to consider the cases in which the solutions have been shown to be impossible, or the second case where B $= 90°$. We shall therefore confine our observations to the third case when a and A are of the same species, and to the fourth case.

(253.) To determine the angle C, we have by [5], Sect. IV.

$$\cos.C \cos.b + \sin.C \cot.A - \sin.b \cot.a = 0,$$

$$\therefore \cos.C + \sin.C \frac{\cot.A}{\cos.b} - \tan.b \cot.a = 0.$$

Let

$$\tan.\theta = \frac{\cot.A}{\cos.b},$$

$$\therefore \frac{\cos.\theta \cos.C + \sin.\theta \sin.C}{\cos.\theta} = \tan.b \cot.a,$$

$$\therefore \cos.(C - \theta) = \tan.b \cot.a \cos.\theta.$$

Since $\cos.(C - \theta) = \cos.(\theta - C)$, it is evident that this equation only determines the difference between C and θ, but does not indicate whether $C > \theta$ or $\theta > C$, and so leaves C doubtful.

From the value of $\tan.\theta$, it appears that b is the hypothenuse of a right-angled triangle, of which A and θ are the oblique angles. Hence it follows that the subsidiary angle θ is the angle which the perpendicular (p) from the angle C upon the side c makes with the side b. The angle under the same perpendicular and the side a is $C - \theta$ if the perpendicular fall inside the triangle, and $\theta - C$ if it fall outside.

It may be proved in general, that the perpendicular falls within or without the triangle according as the angles B and A are of the same or different species. If p be the perpendicular, we have by the two right-angled triangles

$$\cos.A = \cos.p \sin.\theta,$$

$$\cos.B = \cos.p \sin.(C - \theta),$$

$$\therefore \frac{\cos.A}{\cos.B} = \frac{\sin.\theta}{\sin.(C - \theta)}.$$

If A and B be of the same species, $\cos.A$ and $\cos.B$ have the same sign, and therefore $\sin.\theta$ and $\sin.(C - \theta)$ have necessarily the same sign; and since $\sin.\theta$ is positive, $\sin.(C - \theta)$ must be also positive, therefore $C > \theta$, and therefore the perpendicular p falls within the triangle. If A and B be of different species, $\cos.A$ and $\cos.B$ have different signs, and, therefore, $\sin.\theta$ and $\sin.(C - \theta)$ have different signs, and since $\sin.\theta$ is positive, $\sin.(C - \theta)$ is negative, therefore $C < \theta$; and therefore the perpendicular falls without the triangle.

If $\frac{\sin.a}{\sin.b} > 1$, it follows that b and B are of the same species, and therefore the perpendicular will fall within or

without the triangle, according as b and A are of the same or different species.

If φ be the angle whose cosine is

$$\tan.b \cot.a \cos\theta,$$

we have C equal to the sum of φ and θ, when b and A are of the same species, and equal to the difference of φ and θ, when b and A are of different species. In this case, therefore, C can always be determined.

If $\frac{\sin.a}{\sin.b} < 1$, the angle B has two supplemental values, one of which will be of the same, and the other of a different species from A; hence the perpendicular falls within the triangle in the one case, and without it in the other. The angle C is therefore susceptible of either of two values, the sum or difference of φ and θ, since each will satisfy the conditions of the problem.

(254.) To determine the side c, we have

$$\cos.a = \cos.b \cos.c + \sin.b \sin.c \cos.A,$$

$$\therefore \frac{\cos.a}{\cos.b} = \cos.c + \tan.b \sin.c \cos.A.$$

Let

$$\tan.b \cos.A = \tan.\theta,$$

$$\therefore \frac{\cos.a}{\cos.b} = \frac{\cos.c \cos.\theta + \sin.c \sin.\theta}{\cos.\theta}.$$

Let φ be the difference between c and θ. Hence

$$\cos.\varphi = \cos.\theta \frac{\cos.a}{\cos.b}.$$

The side c is the sum or difference of φ and θ.

From the value of $\tan.\theta$ it appears, that b and θ are the hypothenuse and side of a right-angled triangle, and that A is the included angle. Hence this triangle is that which is formed by the side b and the perpendicular p on c. The arc θ is therefore the segment of the base between p and b, and it is plain that φ is the other segment between p and a. Hence it appears that c is equal to the sum or difference of

φ and θ, according as the perpendicular p falls within or without the triangle.

Hence, if $\frac{\sin.a}{\sin.b} > 1$, c is equal to the sum of φ and θ when b and A are of the same species, and to the difference of φ and θ when b and A are of different species.

If $\frac{\sin.a}{\sin.b} < 1$, there are two values of c which equally satisfy the conditions of the problem, one equal to the sum of φ and θ, and the other to their difference.

(255.) If the angle B be previously determined, the included angle C and the side c may be computed by Neper's Analogies.

$$\cot.\tfrac{1}{2}\text{C} = \tan.\tfrac{1}{2}(\text{A} + \text{B})\frac{\cos.\frac{1}{2}(a+b)}{\cos.\frac{1}{2}(a-b)},$$

$$\tan.\tfrac{1}{2}c = \tan.\tfrac{1}{2}(a + b)\frac{\cos.\frac{1}{2}(\text{A}+\text{B})}{\cos.\frac{1}{2}(\text{A}-\text{B})}.$$

In the case where there are two values of B, by substituting them successively for B in these formulæ the corresponding values of C and c may be found. See Table IX.

VI.

Given two angles and the side opposed to one of them.

(256.) Let the angles be A, B, and the side a.

To determine b, we have

$$\sin.b = \frac{\sin.\text{B}}{\sin.\text{A}}\sin.a = \frac{\sin.a}{m}.$$

From reasoning precisely analogous to that used in the last case, and which it is therefore unnecessary to repeat, we can deduce the following conclusions:

1°. If $\frac{\sin.\text{A}}{\sin.\text{B}} < \sin.a$	The problem admits of no solution.

2°. If $\frac{\sin.A}{\sin.B} = \sin.a.$	If a and A be of the same species, $b = 90°$, and there is but one solution (251.). If a and A be of different species, there is no solution.
3°. If $\frac{\sin.A}{\sin.B} > \sin.a$ and < 1	If a and A be of the same species, there are two solutions in which the values of b are supplemental. If a and A be of different species, there is no solution.
4°. If $\frac{\sin.A}{\sin.B}$ not < 1.	There is but one solution, b being of the same species with B.

To determine C and c in the second case, it is only necessary to consider that the supplemental polar triangle is right angled, and that its sides are $\pi - A$ and $\pi - C$, the opposite angles $\pi - a$, $\pi - c$, and the hypothenuse $\pi - B$. Hence the determination of $\pi - C$ and $\pi - c$ is reduced to the third case of right-angled triangles.

(257.) To determine the side c, a formula may be immediately deduced by applying the result of (253.) to the polar triangle, which gives

$$\cot.\theta = \frac{\cot.a}{\cos.B},$$

$$\sin.(c - \theta) = \tan.B \cot.A \sin.\theta.$$

There are two supplemental values of $c - \theta$, which equally satisfy this equation, and therefore the species of $c - \theta$ is not determined. From the value of $\cot.\theta$, it appears that θ is the segment of the side c between the perpendicular and the angle B, and therefore $c - \theta$ must be the other segment.

It is easy to prove that the segments θ and $c - \theta$ are of the same or different species, according as the sides a, b, are of the same or different species; for by Neper's rules,

$$\cos.a = \cos.\theta \cos.p, \quad \cos.b = \cos.(c - \theta)\cos.p,$$

$$\therefore \frac{\cos.a}{\cos.b} = \frac{\cos.\theta}{\cos.(c-\theta)}.$$

Hence $\cos.\theta$ and $\cos.(c - \theta)$ have the same or different si according as $\cos.a$ and $\cos.b$ have the same or diff signs, and therefore θ and $c - \theta$ are of the same o

ferent species, according as a and b are of the same or different species.

If $\frac{\sin.A}{\sin.B} > 1$, it follows that b and B must be of the same species. Hence, in this case θ and $c - \theta$ are of the same species when a and B are of the same species, and θ and $c - \theta$ are of different species when a and B are of different species. In this case, therefore, the value of c is determined.

If $\frac{\sin.A}{\sin.B} < 1$, the side b is susceptible of two supplemental values, one of which is therefore of the same, and the other of a different species from A or a, and, therefore, in this case, either of the two values of $c - \theta$ will indifferently satisfy the conditions of the problem; one of them corresponding to one value of the side b, and the other to the remaining one.

(258.) To determine the angle C, we obtain by the supplemental triangle a formula analogous to that determined in (254.),

$$\cot.\theta = \tan.B \cos.a,$$

$$\frac{\cos.A}{\cos.B} = \frac{\sin.C \cos.\theta - \sin.\theta \cos.C}{\sin.\theta},$$

$$\sin.(C - \theta) = \sin.\theta \frac{\cos.A}{\cos.B}.$$

There are two supplemental values of $(C - \theta)$, which equally satisfy this equation. The species of $(C - \theta)$ remains therefore undetermined. By the value of $\cot.\theta$, it appears that θ is the angle under the perpendicular on the side c from the angle C and the side a. Hence $C - \theta$ is the angle under the same perpendicular and the side b. By Neper's rules applied to the two right-angled triangles, we have

$$\cos.\theta = \cot.a \tan.p,$$

$$\cos.(C - \theta) = \cot.b \tan.p,$$

$$\because \frac{\cos.\theta}{\cos.(c-\theta)} = \frac{\cot.a}{\cot.b},$$

whence it follows that θ and $c - \theta$ are of the same or different species, according as a and b are of the same or different species.

If $\frac{\sin.A}{\sin.B} > 1$, b and B must be of the same species; therefore, in this case $(c - \theta)$ and θ are of the same or different species, according as a and B are of the same or different species. The value of c is therefore in this case determinate.

If $\sin.A < \sin.B$, the side b may have either of two supplemental values, and therefore for the one, $c - \theta$ and θ will be of the same species, and for the other of different species. The two values of $c - \theta$ therefore, in this case, equally satisfy the conditions of the problem, and there are two solutions.

(259.) Those cases of spherical triangles which admit of two solutions, and are thence called the "doubtful cases," may be, in particular instances, rendered determinate by some circumstances connected with the triangle, from which the species of the sought quantity can be known. That which is always to be determined is, whether an angle and the side opposed to it be of the same or different species. To be of different species, it is necessary that the sine of the side should be greater than the sines of either of the other sides, or the sine of the angle greater than the sines of either of the other angles. If any circumstance be known which shows this not to be the case, the problem becomes at once determinate, and the species of the sought quantity is known.

(260.) The obvious analogy which subsists between the formulæ for the solution of plane and spherical triangles cannot but have attracted the notice of the student. Th

origin of this analogy is easily traced. While the absolute lengths of the sides of a spherical triangle remain unchanged, let the radius of the sphere be conceived to be continually increased. The consequence will be, that the angles subtended at the centre of the sphere by the sides will be continually diminished, and therefore the sides expressed in *degrees* will be also continually diminished. If the radius of the sphere be *infinitely* increased, the *finite* portion of its surface filled by the triangle will become a plane, and thus the limiting state of the spherical triangle will be a plane triangle. The effect which this change will produce on the formulæ for the solution of the spherical triangle will be to convert them into the corresponding formulæ for a plane triangle. Hence arises the analogy which may be observed to subsist between the formulæ.

(261.) The formulæ for plane triangles may thus be inferred from those of spherical triangles. But in making the inference, the radius of the sphere, which has been assumed as unity, should be changed to r by the rule in (24.). The sides of the triangle themselves should be substituted for their sines and tangents, on the principle that the limit of the ratios of the sines and tangents to the arcs is a ratio of equality *. The cosines, secants, cotangents, and cosecants, may always be eliminated by the fundamental equations in page 22.

As an example of this process, let it be proposed to deduce from

$$\sin.\text{A} = \frac{2}{\sin.b \sin.c}\sqrt{\sin.s \sin.(s-a)\sin.(s-b)\sin.(s-c)},$$

the corresponding formula for a plane triangle. By the application of the principles already mentioned, the sines being omitted, we have

---

* Differential Calculus, (25.), (26.).

$$\sin.A = \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}.$$

The radius disappears in consequence of entering both numerator and denominator of the second member in the same dimensions (r^2), and sin.A is referred to the radius unity.

(262.) It is sometimes expedient in order to obtain the result to develop the formulæ in series. Thus, let it be required to deduce the corresponding formulæ for plane triangles from

$$\cos.a\cos.b + \sin.a\sin.b\cos.C - \cos.c = 0,$$

$$\cos.a = (1 - \sin.^2a)^{\frac{1}{2}} = 1 - \tfrac{1}{2}\sin.^2a - \tfrac{1}{8}\sin.^4a \cdots\cdots$$

$$\cos.b = (1 - \sin.^2b)^{\frac{1}{2}} = 1 - \tfrac{1}{2}\sin.^2b - \tfrac{1}{8}\sin.^4b \cdots\cdots$$

$$\cos.c = (1 - \sin.^2c)^{\frac{1}{2}} = 1 - \tfrac{1}{2}\sin.^2c - \tfrac{1}{8}\sin.^4c \cdots\cdots$$

$$\because \cos.a\cos.b = 1 - \tfrac{1}{2}(\sin.^2a + \sin.^2b) - \tfrac{1}{8}(\sin.^4a + \sin.^4b) + \cdots\cdots$$

$$\because \cos a\cos b - \cos c = \tfrac{1}{2}(\sin^2c - \sin^2a - \sin^2b) + \tfrac{1}{8}(\sin^4c - \sin^4a - \sin^4b)$$

. . . .

Supplying the radius r by (24.),

$$\cos.a\cos.b - r\cos.c = \tfrac{1}{2}(\sin.^2c - \sin.^2a - \sin.^2b)$$
$$+ \frac{1}{8r^2}(\sin.^4c - \sin.^4a - \sin.^4b) \cdots\cdots$$

which reduces the proposed formula to

$$-2\sin.a\sin.b\cos.C = \sin.^2c - \sin.^2a - \sin.^2b$$
$$+ \frac{1}{4r^2}(\sin.^4c - \sin.^4a - \sin.^4b) + \&c. \cdots\cdots$$

Let r be now supposed infinite, and omitting the sines, the result is

$$2ab\cos.C + c^2 - a^2 - b^2 = 0,$$

$$\because \cos.C = \frac{a^2+b^2-c^2}{2ab},$$

which is the result of (75.).

We shall now investigate the methods of computing the

area of a spherical triangle, being given any three of the six parts, the angles and the sides.

PROP. LXXIV.

(263.) *Given the three angles of a spherical triangle, to compute the area.*

If D be the area,

$$D = r^2(2S - \pi),$$

where $S = \frac{1}{2}(A + B + C)$, and is expressed in relation to the radius unity. If S should be expressed in degrees, and π by 180°, they must both be reduced to seconds, and divided by 206265 (8). In this case $2S - \pi$ will be an abstract number, and the area of the triangle will be determined with reference to (r) the radius of the sphere.

If the radius of the sphere be taken as the unit,

$$D = 2S - \pi.$$

PROP. LXXV.

(264.) *Given the sides of a spherical triangle, to determine the area.*

By the formula

$$D = 2S - \pi,$$

all problems respecting the area D are resolved into equivalent problems respecting S. By the equations

$$\sin.\tfrac{1}{2}D = -\cos.S,$$

$$\cos.S = \cos.\tfrac{1}{2}(A + B)\cos.\tfrac{1}{2}C - \sin.\tfrac{1}{2}(A + B)\sin.\tfrac{1}{2}C,$$

$$\cos.\tfrac{1}{2}(A + B) = \frac{\sin.\frac{1}{2}c}{\cos.\frac{1}{2}c}\cos.\tfrac{1}{2}(a + b),$$

$$\sin.\tfrac{1}{2}(A + B) = \frac{\cos.\frac{1}{2}C}{\cos.\frac{1}{2}c}\cos.\tfrac{1}{2}(a - b),$$

we find

$$\sin.\tfrac{1}{2}D = \frac{\sin.\tfrac{1}{2}a\sin.\tfrac{1}{2}b}{\cos.\tfrac{1}{2}c}\sin.C.$$

If, besides the three sides an angle be given, this formula will resolve the problem. Otherwise, substituting for sin.C its value in terms of the sides (182.), [2], we obtain

$$\sin.\tfrac{1}{2}D = \frac{\sqrt{\sin.s\sin.(s-a)\sin.(s-b)\sin.(s-c)}}{2\cos.\tfrac{1}{2}a\cos.\tfrac{1}{2}b\cos.\tfrac{1}{2}c}.$$

which is a symmetrical function of the sides, and suited to logarithms.

Other expressions remarkable for their symmetry may be obtained for the area in terms of the sides. By the equation

$$\cos.\tfrac{1}{2}D = \sin.S = \sin.\tfrac{1}{2}(A+B)\cos.\tfrac{1}{2}C + \sin.\tfrac{1}{2}C\cos.\tfrac{1}{2}(A+B),$$

we obtain, as in the last case,

$$\cos.\tfrac{1}{2}D = \frac{\cos.\tfrac{1}{2}(a-b)\cos.^2\tfrac{1}{2}C + \cos.\tfrac{1}{2}(a+b)\sin.^2\tfrac{1}{2}C}{\cos.\tfrac{1}{2}c},$$

$$\therefore \cos.\tfrac{1}{2}D = \frac{\cos.\tfrac{1}{2}(a-b) + [\cos.\tfrac{1}{2}(a+b) - \cos.\tfrac{1}{2}(a-b)]\sin.^2\tfrac{1}{2}C}{\cos.\tfrac{1}{2}c},$$

$$\therefore \cos.\tfrac{1}{2}D = \frac{\cos.\tfrac{1}{2}(a-b) - 2\sin.\tfrac{1}{2}a\sin.\tfrac{1}{2}b\sin.^2\tfrac{1}{2}C}{\cos.\tfrac{1}{2}c},$$

$$= \frac{\cos.\tfrac{1}{2}a\cos.\tfrac{1}{2}b + \sin.\tfrac{1}{2}a\sin.\tfrac{1}{2}b(1-2\sin.^2\tfrac{1}{2}C)}{\cos.\tfrac{1}{2}c},$$

$$= \frac{\cos.\tfrac{1}{2}a\cos.\tfrac{1}{2}b + \sin.\tfrac{1}{2}a\sin.\tfrac{1}{2}b\cos.C}{\cos.\tfrac{1}{2}c}.$$

This last value will determine the area by $\cos.\tfrac{1}{2}D$ when the angle C is given with the sides.

From the third of the preceding values by eliminating $\sin.^2\tfrac{1}{2}C$ by its value in (222.), we obtain

$$\cos.\tfrac{1}{2}D = \frac{\cos.\tfrac{1}{2}(a-b)}{\cos.\tfrac{1}{2}c} - \frac{\sin.(s-a)\sin.(s-b)}{2\cos.\tfrac{1}{2}a\cos.\tfrac{1}{2}b\cos.\tfrac{1}{2}c}.$$

But

$$\sin.(s-a)\sin.(s-b) = \tfrac{1}{2}[\cos.(b-a) - \cos.c].$$

Hence

$$\cos.\tfrac{1}{2}D = \frac{2\cos.\tfrac{1}{2}a\cos.\tfrac{1}{2}b\cos.\tfrac{1}{2}(a-b) - \tfrac{1}{2}\cos.(a-b) + \tfrac{1}{2}\cos.c}{2\cos.\tfrac{1}{2}a\cos.\tfrac{1}{2}b\cos.\tfrac{1}{2}c}.$$

Eliminating $\cos.(a-b)$ by

$$\cos.(a-b) = 2\cos.^2\tfrac{1}{2}(a-b) - 1,$$

and observing that

$$\cos.\tfrac{1}{2}(a+b)\cos.\tfrac{1}{2}(a-b) + \tfrac{1}{2}\cos.c + \tfrac{1}{2} = \frac{\cos.a+\cos.b+\cos.c+1}{2},$$

we obtain

$$\cos.\tfrac{1}{2}D = \frac{\cos.a+\cos.b+\cos.c+1}{4\cos.\tfrac{1}{2}a\cos.\tfrac{1}{2}b\cos.\tfrac{1}{2}c}$$

$$= \frac{\cos.^2\tfrac{1}{2}a+\cos.^2\tfrac{1}{2}b+\cos.^2\tfrac{1}{2}c-1}{2\cos.\tfrac{1}{2}a\cos.\tfrac{1}{2}b\cos.\tfrac{1}{2}c},$$

a symmetrical formula for $\cos.\tfrac{1}{2}D$ in terms of the sides.

By combining the results already obtained, we shall obtain a formula of $\tan.\tfrac{1}{4}D$ of singular beauty;

$$\tan.\tfrac{1}{4}D = \frac{1-\cos.\tfrac{1}{2}D}{\sin.\tfrac{1}{2}D},$$

which after substituting the values already obtained for $\sin.\tfrac{1}{2}D$ and $\cos.\tfrac{1}{2}D$, becomes

$$\tan.\tfrac{1}{4}D = \frac{1-\cos.^2\tfrac{1}{2}a-\cos.^2\tfrac{1}{2}b-\cos.^2\tfrac{1}{2}c+2\cos.\tfrac{1}{2}a\cos.\tfrac{1}{2}b\cos.\tfrac{1}{2}c}{\sqrt{\sin.s\sin.(s-a)\sin.(s-b)\sin.(s-c)}}.$$

The numerator of this is equivalent to

$$\sin.^2\tfrac{1}{2}a\sin.^2\tfrac{1}{2}b - [\cos.\tfrac{1}{2}a\cos.\tfrac{1}{2}b - \cos.\tfrac{1}{2}c]^2,$$

which is equal to

$$\begin{aligned}&[\sin.\tfrac{1}{2}a\sin.\tfrac{1}{2}b + \cos.\tfrac{1}{2}a\cos.\tfrac{1}{2}b - \cos.\tfrac{1}{2}c]\\ \times&[\sin.\tfrac{1}{2}a\sin.\tfrac{1}{2}b - \cos\tfrac{1}{2}a\cos.\tfrac{1}{2}b + \cos.\tfrac{1}{2}c] =\\ &[\cos.\tfrac{1}{2}(a-b)-\cos.\tfrac{1}{2}c]\times[\cos.\tfrac{1}{2}c-\cos.\tfrac{1}{2}(a+b)] =\\ &4\sin.\tfrac{1}{2}s\sin.\tfrac{1}{2}(s-a)\sin.\tfrac{1}{2}(s-b)\sin.\tfrac{1}{2}(s-c).\end{aligned}$$

Hence we find

$$\tan.\tfrac{1}{4}D = \sqrt{\tan.\tfrac{1}{2}s\tan.\tfrac{1}{2}(s-a)\tan.\tfrac{1}{2}(s-b)\tan.\tfrac{1}{2}(s-c)}.$$

This beautiful formula was first obtained by *Lhuilier* of *Geneva*.

(265.) The formula (77.) for the area of a plane triangle

follows from this by omitting the tangents and retaining only the sides, which gives

$$\tfrac{1}{4}\text{D} = \sqrt{\tfrac{1}{2}s \cdot \tfrac{1}{2}(s-a) \cdot \tfrac{1}{2}(s-b) \cdot \tfrac{1}{2}(s-c)},$$

$$\therefore \text{D} = \sqrt{s(s-a)(s-b)(s-c)}.$$

PROP. LXXVI.

(266.) *Given two sides of a spherical triangle and the included angle, to compute the area.*

By division, from

$$\sin.\tfrac{1}{2}\text{D} = \frac{\sin.\frac{1}{2}a \sin.\frac{1}{2}b}{\cos.\frac{1}{2}c}\sin.\text{C}.$$

$$\cos.\tfrac{1}{2}\text{D} = \frac{\cos.\frac{1}{2}a \cos.\frac{1}{2}b + \sin.\frac{1}{2}a \sin.\frac{1}{2}b \cos.\text{C}}{\cos.\frac{1}{2}c},$$

obtained in the last proposition, we obtain

$$\tan.\tfrac{1}{2}\text{D} = \frac{\sin.\frac{1}{2}a \sin.\frac{1}{2}b \sin.\text{C}}{\cos.\frac{1}{2}a \cos.\frac{1}{2}b + \sin.\frac{1}{2}a \sin.\frac{1}{2}b \cos.\text{C}},$$

$$\therefore \tan.\tfrac{1}{2}\text{D} = \frac{\tan.\frac{1}{2}a \tan.\frac{1}{2}b \sin.\text{C}}{1 + \tan.\frac{1}{2}a \tan.\frac{1}{2}b \cos.\text{C}}.$$

This may be adapted for logarithms by means of subsidiary angles. Let $2\cos.^2\frac{1}{2}\text{C} - 1$ be substituted for $\cos.\text{C}$ in the former value, and we find

$$\tan.\tfrac{1}{2}\text{D} = \frac{\sin.\frac{1}{2}a \sin.\frac{1}{2}b \sin.\text{C}}{\cos.\frac{1}{2}(a+b) + 2\sin.\frac{1}{2}a \sin.\frac{1}{2}b \cos.^2\frac{1}{2}\text{C}}.$$

Now let

$$\sin.\tfrac{1}{2}a \sin.\tfrac{1}{2}b \cos.^2\tfrac{1}{2}\text{C} = \cos.\theta,$$

$$\cos.\tfrac{1}{2}(a+b) = 2\cos.\phi,$$

which conditions can always be fulfilled. Hence

$$\cos.\tfrac{1}{2}(a+b) + 2\sin.\tfrac{1}{2}a \sin.\tfrac{1}{2}b \cos.^2\tfrac{1}{2}\text{C} = 2(\cos.\phi + \cos.\theta)$$

$$\cos.\phi + \cos.\theta = 2\cos.\tfrac{1}{2}(\phi+\theta)\cos.\tfrac{1}{2}(\phi-\theta).$$

Hence

$$\tan.\tfrac{1}{2}\text{D} = \frac{\sin.\frac{1}{2}a \sin.\frac{1}{2}b \sin.\text{C}}{4\cos.\frac{1}{2}(\phi+\theta)\cos.\frac{1}{2}(\phi-\theta)},$$

which is suited for logarithms.

SECTION VIII.

Examples on oblique-angled spherical triangles.

PROP. LXXVII.

(267.) *To determine the relation between the segments* (α, β) *of the base* (c) *of a spherical triangle made by the arc* (φ) *of a great circle bisecting the vertical angle* (C).

Let the angle at which φ is inclined to the base be ψ. Hence

$$\frac{\sin.\alpha}{\sin.a}=\frac{\sin.\frac{1}{2}C}{\sin.\psi}, \quad \frac{\sin.\beta}{\sin.b}=\frac{\sin.\frac{1}{2}C}{\sin.\psi},$$

$$\therefore \frac{\sin.\alpha}{\sin.\beta}=\frac{\sin.a}{\sin.b};$$

that is, the bisector of the vertical angle divides the base into segments, whose sines are proportional to those of the conterminous sides.

It is obvious that Eucl. lib. vi. prop. iii. is included in this result (260.).

(268.) *Cor.* 1. Hence it follows that the three bisectors of the angles intersect at the same point. For the bisector of the angle A must divide φ into segments whose sines are as $\sin.\alpha : \sin.a$, and the bisector of B must divide it into segments whose sines are as $\sin.\beta : \sin.b$. But by this proposition,

$$\frac{\sin.\alpha}{\sin.a}=\frac{\sin.\beta}{\sin.b}.$$

Hence the bisectors of A and B must meet φ at the same point.

(269.) *Cor.* 2. The segments α and β may be easily computed,

$$\frac{\sin.\alpha+\sin.\beta}{\sin.\alpha-\sin.\beta}=\frac{\sin.a+\sin.b}{\sin.a-\sin.b}=\frac{\tan.\frac{1}{2}(a+b)}{\tan.\frac{1}{2}(a-b)},$$

$$\therefore \tan.\tfrac{1}{2}(\alpha-\beta)=\tan.\tfrac{1}{2}(\alpha+\beta)\,\frac{\tan.\frac{1}{2}(a-b)}{\tan.\frac{1}{2}(a+b)},$$

$$\therefore \tan.\tfrac{1}{2}(\alpha-\beta)=\tan.\tfrac{1}{2}c\frac{\tan.\frac{1}{2}(a-b)}{\tan.\frac{1}{2}(a+b)}.$$

Whence $\alpha-\beta$ being found, α and β may be determined.

(270.) *Cor.* 3. It is evident, for the same reasons as in plane triangles, that if perpendiculars be drawn from the intersection of the bisectors on the sides they will be equal, and thus "the intersection of the bisectors is the pole of the lesser circle inscribed in the triangle touching the three sides."

The circular radius of this circle or the common length of the perpendiculars is easily determined. It is obvious that the parts of two sides between their intersection and the radius are equal. Hence it follows that the two segments into which c is divided by the radius, are

$$s-a, \quad s-b.$$

Hence (r = radius),

$$\begin{aligned}\tan.r &= \tan.\tfrac{1}{2}\text{A}\,\sin.(s-a)\\ &= \tan.\tfrac{1}{2}\text{B}\,\sin.(s-b)\\ &= \tan.\tfrac{1}{2}\text{C}\,\sin.(s-c).\end{aligned}$$

The radius being symmetrically related to the sides and angles, should be a symmetrical function of them. None of these values just obtained are so; but we may find one by multiplying the three together, which gives

$$\tan.^3r=\tan.\tfrac{1}{2}\text{A}\,\tan.\tfrac{1}{2}\text{B}\,\tan.\tfrac{1}{2}\text{C}\,\sin.(s-a)\sin.(s-b)\sin.(s-c).$$

This is a symmetrical function of the sides and angles. We may find a symmetrical function of the sides alone by eliminating the product of the tangents by its value in 58, Table VIII., $\therefore$

$$\tan.^3 r = \frac{[\sin.(s-a)\sin.(s-b)\sin.(s-c)]^{\frac{3}{2}}}{\sin.^{\frac{3}{2}} s},$$

$$\because \tan.r = \frac{\sqrt{\sin.(s-a)\sin.(s-b)\sin.(s-c)}}{\sqrt{\sin.s}} = \frac{n}{\sin.s}$$

$$= \frac{\mathrm{N}}{2\cos.\frac{1}{2}\mathrm{A}\cos.\frac{1}{2}\mathrm{B}\cos.\frac{1}{2}\mathrm{C}}.$$

If each pair of sides of the triangle be produced until they intersect, three other triangles will be found, of which the circular radius of the inscribed circle may be found similarly. Let a and b be produced through c until they intersect. The sides of the triangle formed by their productions will be (131.),

$$a' = \pi - a, \quad b' = \pi - b, \quad c' = c,$$

and its angles

$$\mathrm{A}' = \pi - \mathrm{A}, \quad \mathrm{B}' = \pi - \mathrm{B}, \quad \mathrm{C}' = \mathrm{C}.$$

Hence

$$s' = \pi - (s - c), \quad s' - a' = s - b,$$

$$s' - b' = s - a, \quad s' - c' = \pi - s,$$

$$\because \sin.s' \sin.(s' - a') \sin.(s' - b') \sin.(s' - c')$$

$$= \sin.s \sin.(s - a) \sin.(s - b) \sin.(s - c).$$

Hence, if r' be the radius of the inscribed circle,

$$\tan.r' = \frac{n}{\sin.(s-c)},$$

and in like manner the radii of the circles inscribed in the other two triangles are

$$\tan.r'' = \frac{n}{\sin.(s-a)},$$

$$\tan.r''' = \frac{n}{\sin.(s-b)}.$$

Hence it follows that

$$\tan.r \tan.r' \tan.r'' \tan.r''' = n^2.$$

PROP. LXXVIII.

(271.) *To determine the pole and the radius of the circle which circumscribes a spherical triangle.*

Let R be the radius. The three radii from the pole to the angles form three isosceles triangles. Let the base angle of that isosceles triangle, whose base is the side a, be α; and, in like manner, let β and γ be the base angles of those whose bases are b and c respectively. It is evident that

$$\alpha + \beta = \text{C},$$
$$\beta + \gamma = \text{A},$$
$$\alpha + \gamma = \text{B}.$$

Hence we find

$$\alpha = \tfrac{1}{2}(\text{B} + \text{C} - \text{A}) = \text{S} - \text{A},$$
$$\beta = \tfrac{1}{2}(\text{A} + \text{C} - \text{B}) = \text{S} - \text{B},$$
$$\gamma = \tfrac{1}{2}(\text{A} + \text{B} - \text{C}) = \text{S} - \text{C},$$

where $\text{S} = \tfrac{1}{2}(\text{A} + \text{B} + \text{C})$.

By Tab. VIII. we find

$$\tan.\alpha = \frac{1 + \cos.a - \cos.b - \cos.c}{2n},$$

$$\tan.\beta = \frac{1 + \cos.b - \cos.a - \cos.c}{2n},$$

$$\tan.\gamma = \frac{1 + \cos.c - \cos.a - \cos.b}{2n},$$

where $n = \sqrt{\sin.s \sin.(s-a) \sin.(s-b) \sin.(s-c)}$.

When the triangle is given, any one of the angles α, β, and γ, are sufficient to determine the pole of the circle circumscribing it.

To determine the radius R. If a perpendicular be supposed to be drawn from the pole to the side a, it will divide the isosceles triangle into two equal right-[illegible] triangles, which give

$$\tan.\text{R} = \frac{\tan.\tfrac{1}{2}a}{\cos.[illegible]}$$

and in like manner by the other isosceles triangles, we have

$$\tan.\text{R} = \frac{\tan.\frac{1}{2}b}{\cos.\beta},$$

$$\tan.\text{R} = \frac{\tan.\frac{1}{2}c}{\cos.\gamma}.$$

Any of these would be sufficient to determine R; α, β, or γ, having been previously found. However, it is desirable that R should be determined immediately as a function of the sides, and this function should be symmetrical.

Let the three values of tan.R just found be multiplied,

$$\tan.^3\text{R} = \frac{\tan.\frac{1}{2}a \tan.\frac{1}{2}b \tan.\frac{1}{2}c}{\cos.\alpha \cos.\beta \cos.\gamma}.$$

But by the results of Tab. VIII.,

$$\cos.\alpha = \frac{\cos.\frac{1}{2}b \cos.\frac{1}{2}c \sin.\text{A}}{\cos.\frac{1}{2}a},$$

$$\cos.\beta = \frac{\cos.\frac{1}{2}a \cos.\frac{1}{2}c \sin.\text{B}}{\cos.\frac{1}{2}b},$$

$$\cos.\gamma = \frac{\cos.\frac{1}{2}a \cos.\frac{1}{2}b \sin.\text{C}}{\cos.\frac{1}{2}c}.$$

Hence

$$\cos.\alpha \cos.\beta \cos.\gamma = \cos.\tfrac{1}{2}a \cos.\tfrac{1}{2}b \cos.\tfrac{1}{2}c \sin.\text{A} \sin.\text{B} \sin.\text{C}.$$

Substituting for sin.A. sin.B. sin.C, its value in (199.), [6], we obtain

$$\cos.\alpha \cos.\beta \cos.\gamma = \frac{8n^3 \cos.\frac{1}{2}a \cos.\frac{1}{2}b \cos.\frac{1}{2}c}{\sin.^2 a \sin.^2 b \sin.^2 c}.$$

Hence

$$8n^3 \tan.^3\text{R} = \frac{\sin.\frac{1}{2}a \sin.\frac{1}{2}b \sin.\frac{1}{2}c \sin.^2 a \sin.^2 b \sin.^2 c}{\cos.^2\frac{1}{2}a \cos.^2\frac{1}{2}b \cos.^2\frac{1}{2}c}$$

$$= 4^3 \sin.^3\tfrac{1}{2}a \sin.^3\tfrac{1}{2}b \sin.^3\tfrac{1}{2}c,$$

$$\therefore \tan.\text{R} = \frac{2\sin.\frac{1}{2}a \sin.\frac{1}{2}b \sin.\frac{1}{2}c}{n},$$

a formula of remarkable symmetry.

From this formula, by omitting sines (260.), we may immediately deduce the well known formula for the radius of the circle circumscribing a plane triangle,

$$\text{R} = \frac{\frac{1}{2}abc}{2\sqrt{s(s-a)(s-b)(s-c)}} = \frac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}}.$$

The value of cot.R may be reduced to a function of the angles by [23], Sect. IV., which gives

$$\cot.\text{R} = -\frac{\text{N}}{\cos.\text{S}}.$$

The radii of the circles circumscribed round the three triangles formed by producing every pair of the sides may be found by the substitutions used in the last proposition. These give, as in the last proposition,

$$\cot.\text{R}' = -\frac{\text{N}}{\cos.(\text{S}-\text{C})},$$

$$\cot.\text{R}'' = -\frac{\text{N}}{\cos.(\text{S}-\text{A})},$$

$$\cot.\text{R}''' = -\frac{\text{N}}{\cos.(\text{S}-\text{B})}.$$

Hence we find

$$\cot.\text{R}\cot.\text{R}'\cot.\text{R}''\cot.\text{R}''' = \text{N}^2.$$

PROP. LXXIX.

(272.) *Given the base and the area of a spherical triangle, to determine the locus of its vertex.*

Since the area is given, the sum of its three angles is given (177.). If a be the given base, and the sides b, c, be produced through it to intersect, a triangle will be formed whose sides and angles are

$$a' = a, \quad b' = \pi - b, \quad c' = \pi - c,$$

$$\text{A}' = \text{A}, \quad \text{B}' = \pi - \text{B}, \quad \text{C}' = \pi - \text{C}.$$

The circular radius R′ of the circle circumscribing this is the last proposition) determined by

$$\tan.R' = \frac{\tan.\frac{1}{2}a'}{\cos.(s'-A')}.$$

But since $a' = a$, and

$$s' - A' = \pi - \frac{B+C-A}{2} - A = \pi - s,$$

$$\therefore \tan.R' = -\frac{\tan.\frac{1}{2}a}{\cos.s},$$

which, since s and a are given, is constant.

The circle circumscribed round the triangle $a'b'c'$ is therefore given in magnitude, but since it passes through the extremities of the given base a, it is also given in position, and is therefore the locus of the vertex of the angle A'. If through this vertex a diameter of the sphere be drawn, its opposite extremity will be in the vertex of A, and therefore its locus must be a lesser circle equal and parallel to the former, and whose pole is the opposite extremity of the diameter.

This locus may be easily constructed. Let the sides b, c, of the given triangle be produced through the vertex of A until they meet the great circle of which the base a forms a part. The triangle thus formed will have its three vertices *diametrically* opposed to those of the other triangle formed by producing b, c, through the base a. Hence the locus is the lesser circle circumscribing this triangle. Let any two of its sides be bisected, and perpendiculars drawn through their middle points until they meet, their point of intersection will be the pole of the lesser circle sought.

This beautiful theorem was discovered by *Lexell*, and published in the first part of the fifth volume of the Petersburgh Acts.

Legendre has given a demonstration of it in his Geometry, note x. It is different, however, from the above.

PROP. LXXX.

(273.) *Given the vertical angle in magnitude and position and the perimeter of a spherical triangle, to determine the curve to which the base is always a tangent.*

By (270.) the radius of the circle inscribed in the triangle formed by producing the sides b, c, which contain the given angle through the base, is determined by

$$\tan.r = \tan.\tfrac{1}{2}A'\sin.(s' - a');$$

but $A' = A$, and $a' = a$, and $s' = \pi - \dfrac{b+c-a}{2} = \pi - (s - a)$.

Hence

$$s' - a' = \pi - s,$$

$$\therefore \tan.r = \tan.\tfrac{1}{2}A\sin.s,$$

which is constant. Hence the bases all touch the lesser circle, which is drawn with this circular radius touching the sides of the given angle A.

PROP. LXXXI.

(274.) *Given two sides of a spherical triangle, to determine the condition on which the area will be a maximum.*

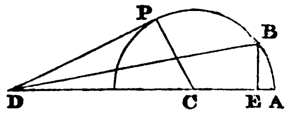
By (266.),

$$\tan.\tfrac{1}{2}D = \frac{\tan.\frac{1}{2}a\tan.\frac{1}{2}b\sin.C}{1+\tan.\frac{1}{2}a\tan.\frac{1}{2}b\cos.C},$$

$$\therefore \cot.\tfrac{1}{2}D = \frac{\cot.\frac{1}{2}a\cot.\frac{1}{2}b+\cos.C}{\sin.C}.$$

The maximum value of (D) might easily be determined by the differential calculus from this formula. We shall, however, give here a beautiful construction by which Legendre determines it.

Let $AB = C$, $\therefore BE = \sin.C$, $CE = \cos.C$.



Let $\text{CD} = \cot.\frac{1}{2}a \cot.\frac{1}{2}b$, $\therefore$ $\frac{\text{DE}}{\text{BE}} = \cot.\text{BDE} = \cot.\frac{1}{2}\text{D}$. Hence D is a maximum when the angle BDE is so. This takes place when DB becomes a tangent DP. Hence, since

$$\text{PCA} - \frac{\pi}{2} = \text{PDC},$$

$$\therefore \text{C} - \frac{\pi}{2} = \tfrac{1}{2}\text{D} = \tfrac{1}{2}(\text{A}+\text{B}+\text{C}) - \frac{\pi}{2},$$

$$\therefore \text{C} = \text{A} + \text{B}.$$

Hence "the area is a maximum when the angle included by the given sides is equal to the sum of the two remaining angles." It is obvious that the value of C is determined by

$$\sec.(\pi - \text{C}) = \cot.\tfrac{1}{2}a \cot.\tfrac{1}{2}b.$$

This result becomes impossible when $\cot.\frac{1}{2}a \cot.\frac{1}{2}b < 1$, that is, when

$$\cot.\tfrac{1}{2}a < \tan.\tfrac{1}{2}b,$$

$$\cot.\tfrac{1}{2}a < \cot.\left(\frac{\pi}{2} - \tfrac{1}{2}b\right),$$

$$\therefore \tfrac{1}{2}a > \frac{\pi}{2} - \tfrac{1}{2}b,$$

$$\therefore a + b > \pi.$$

If the sum of the given sides exceed a semicircle, there will be therefore no maximum. The reason of which is, that, as the angle C augments, the area continually increases until $\text{C} = \pi$, and in that case the two sides are parts of the same great circle, and the third side is the remaining part, so that the triangle becomes an hemisphere, and ceases to be a triangle. (See *Legendre, Geometrie,* liv. vii. prop. xxvi. and note x.).

PROP. LXXXII.

(275.) *Given the latitudes and longitudes of two places, to compute the distance between them.*

The latitudes being given, the distances from the pole, in degrees, may be found. These, with the difference of their longitudes, reduce this problem to the third case of the solution of oblique spherical triangles, page 123.

PROP. LXXXIII.

(276.) *Given the latitudes of two places and the distance between them, to find the difference of their longitudes.*

In a similar way this is reduced to the first case of the solution of oblique spherical triangles, page 119. This problem is of considerable use in navigation. The distance sailed over being known by reckoning, and the latitudes being easily observed. The distance should, however, be reduced to degrees.

PROP. LXXXIV.

(277.) *Given the right ascension and declination of a star, to compute its latitude and longitude, and the angle of position* *.

Let arcs of great circles be imagined to be drawn from the star to the poles of the ecliptic and equator. There will thus be a triangle formed, two sides of which will be the complements of the declination and the latitude. The third side, being the distance between the poles, will be equal to the obliquity of the ecliptic, which is supposed to be known.

---

* The examples which involve astronomical terms may be omitted until after the student has become acquainted with the elements of that science. Little more than its definitions, however, are necessary to render the examples given here intelligible.

The angle at the pole of the ecliptic is equal to the difference between the longitude and a right angle; and that at the pole of the equator is equal to the difference between the right ascension and a right angle. The angle at the star is the angle of position. Thus it appears that two sides and the included angle are given to find the other parts, and the problem is therefore reduced to the third case of oblique spherical triangles.

If L = lat., l = lon., R = right ascen., D = dec., and O = obl. P = angle of position, the problem may be solved by (230.) by the following substitutions:

$$a = \text{O}, \quad b = \frac{\pi}{2} - \text{D}, \quad c = \frac{\pi}{2} - \text{L},$$

$$\text{A} = \text{P}, \quad \text{B} = \left(\frac{\pi}{2} - l\right), \text{C} = \pm\left(\frac{\pi}{2} - \text{R}\right).$$

PROP. LXXXV.

(278.) *Given the declinations and the difference of the right ascensions of two stars, to compute the angular distance between them.*

The declinations being known, the polar distances which are their complements may be found, and the problem is thus reduced to the third case of Sect. VII.

SECTION IX.

On the relations between the small variations in the sides and angles of triangles *.

(279.) We have already shown, that in all determinate

* In order to study the subject of this section with advantage, the first fifty-six articles of my Differential Calculus, or the corresponding parts of some other work on that subject, should be read.

problems respecting the solution of triangles, it is indispensably necessary that three of the six parts of the triangle should be known, and in plane triangles, one at least of the three must be a side. In practice, these data, always obtained originally by observation and measurement, are liable to error from obvious and inevitable causes. It is true that, from the great excellence of instruments, and the almost inconceivable accuracy of modern observation, these errors are extremely minute, yet, in cases where great precision is requisite, it becomes necessary to determine the effects which small errors in the data will produce upon the computed quantities, and to select the data and quæsita in such a manner, that given errors in the one shall entail upon the other the smallest possible errors.

The principles of the differential calculus present easy means for attaining this end. Let us suppose that of the three data, two have been obtained with sufficient accuracy, but the third, x, is liable to an error of a given amount, which we shall call h. Let u be the sought quantity. Two of the three data being considered constant, the sought quantity u may be considered as a function of the third, x, so that

$$u = \mathrm{F}(x).$$

The quantity x becoming $x + h$, let the quantity u become u', we have *

$$u' = \mathrm{F}(x + h)$$

$$\therefore u' - u = \mathrm{A}_1 \frac{h}{1} + \mathrm{A}_2 . \frac{h^2}{1.2} + \mathrm{A}_3 \frac{h^3}{1.2.3} + \cdots$$

where A_1, A_2, A_3, ... are the successive differential coefficients of $\mathrm{F}(x)$.

If x be supposed to represent the true value of that part

* Differential and Integral Calculus (51.), (52.).

of the triangle which is liable to the error h, then $x + h$ will be the quantity given by observation, and u will be the true value of the sought quantity, and u' its computed value. Hence $u' - u$ is the error sought, which is therefore represented by the above series. Since h in practice is always a very small quantity, this series converges rapidly, and therefore a small number of its initial terms may be assumed as equivalent to the whole without sensible error. The number of terms to be taken for the whole depends entirely on the magnitude of the error h. In most cases it is sufficient to take the first term only, but in case h be not extremely small, or if more than ordinary accuracy be requisite, the first two terms are taken; this, however, is seldom necessary, so that we may in general assume

$$u' - u = \text{A}_1 h,$$

which gives the following rule for determining the error in a computed quantity produced by a small error one of the given quantities from whence it is derived:

Let the computed quantity be expressed as a function of the given quantity, and let the differential coefficient be found with respect to the given quantity as a variable, the error in the computed quantity will be determined by multiplying the error in the given quantity by this differential coefficient.

(280.) If two of the data be liable to given errors, the effect upon the sought quantity may be computed on similar principles, by considering the sought quantity as a function of the two data so liable to error, and differentiating it with respect to these as two independent variables *; the differential of the sought quantity thus found will represent the error to which it is liable, the differentials of the data representing their respective errors which are supposed to be very small and given.

It is evident that the same method extends to the case

* Differential Calculus, Sect. VIII.

where all the data are liable to given small errors. In this case the sought quantity is to be regarded as a function of three variables, and its differential found as before.

(281.) The principles which have just been established furnish a method by which, when a triangle plane or spherical is subject to minute variations in its sides or angles, the relation between these variations and the sides and angles themselves may always be investigated and expressed by equations, so that when there are sufficient data, any one of the variations may be derived from the others.

The investigation of the limits of error in the solution of triangles is not the only useful application of these principles in physical science. There are, particularly in astronomy, certain small quantities called *corrections*, the values of which are known, or may be found, and which produce small changes in the magnitudes of quantities engaged in the solution of problems. It is frequently necessary to trace the effects of these on other quantities, whose values result by computation from the former. This may always be done by considering the quantities which these corrections immediately affect, as well as those to be computed from them as variables, and having differentiated the equation or equations which express their relations, considering the several differentials as the corrections and their effects upon the computed quantities.

An example will render this easily understood. It is known that light in passing from a visible object in the heavens, as a star, is deflected from its rectilinear course in such a manner, that it enters the eye of a spectator as if it came from a different point in the heavens, and the spectator sees the star as if it were really placed at this point, and the light came straight from it. This effect, which is called refraction, is such, that the *apparent* place of the star is on the same vertical circle with its true place, but nearer to the

zenith by a very small angle. The magnitude of this angle is known by the tables of refraction. Now if it be required to ascertain the effect of the refraction of a star at a given zenith distance upon its polar distance, it amounts to no more than this; in the spherical triangle formed by the zenith, the pole, and the star, to determine the effect which a small variation in one side produces upon the other, the third side being constant.

The same principle is applicable to the computation of the effects of precession, nutation, parallax, aberration, and various other small *inequalities*, as they are technically called, upon the elements of the position of a celestial object.

The following examples will illustrate the application of this principle.

PROP. LXXXVI.

(282.) *To determine the relation between the minute variations of the side of a plane right-angled triangle and the opposite angle, the remaining side being considered constant.*

Let a and A be the side and angle which are subject to variation, and b the constant side.

$$a = b\tan.\text{A},$$

$$\therefore da = b\sec.^2\text{A}d\text{A},$$

which is the variation sought. To determine this for any given value of a, let b be eliminated by the two equations, and the result is

$$da = a(\cot.\text{A} + \tan.\text{A})d\text{A}.$$

$$= \frac{2a}{\sin.2\text{A}}d\text{A}.$$

(283.) *Cor.* Hence, for any proposed values of a and $d\text{A}$, da is a minimum when $\text{A} = 45°$. Thus, if the height of an object is to be computed by knowing the distance from its

base, and the angle of elevation of its summit, a given error in the elevation will produce a less effect upon the computed height the nearer the elevation is to 45°.

PROP. LXXXVII.

(284.) *Two sides of a plane triangle being given, to investigate the relation between the small variations of the included angle and the opposite side.*

Let a and b be the given sides, and C the included angle, $\therefore$

$$c^2 = a^2 + b^2 - 2ab\cos.\mathrm{C},$$
$$\therefore cdc = ab\sin.\mathrm{C}d\mathrm{C}.$$

But $\frac{1}{2}ab\sin.\mathrm{C}$ being the area of the triangle (77.), if p be the perpendicular from C upon c, we have

$$pc = ab\sin.\mathrm{C},$$
$$\therefore dc = pd\mathrm{C},$$

which is the relation required.

(285.) *Cor.* Hence, if the angle be required to be computed from the side c, a given error in c will produce the least effect on C when the angle opposite to the greater of the given sides a, b, is a right angle. For then p is a maximum.

If, on the other hand, the angle C be given to compute the side c, a given error in C will produce a less effect upon c as the angle C approaches either to zero or 180°. In the one case the limiting value of c is $a - b$, and in the other $a + b$.

PROP. LXXXVIII.

(286.) *One side of a plane triangle being given, it is required to determine the relation between the small variations in the angles and the remaining sides.*

Let a be the given side, and let the equation

$$a\sin.\text{B} = b\sin.\text{A}$$

be differentiated, a alone being considered constant. Hence

$$a\cos.\text{B}d\text{B} = b\cos.\text{A}d\text{A} + \sin.\text{A}db,$$

$$\therefore db + b(\cot.\text{A}d\text{A} - \cot.\text{B}d\text{B}) = 0,$$

a being eliminated by the first equation. Either of the last equations expresses the required relation.

(287.) *Cor.* If the side b be required to be computed from the side a given accurately, and the angles A and B observed subject to small errors, the above equation gives the error of b corresponding to any proposed value of it, for

$$db = b(\cot.\text{B}d\text{B} - \cot.\text{A}d\text{A}).$$

If the errors $d\text{A}$ and $d\text{B}$ be supposed to be equal, and have the same sign (as they probably will, if observed with the same instrument and in the same manner), that is, if the observed angles be both greater or both less than the true by the same quantity, we have

$$d\text{B} = d\text{A},$$

$$\therefore db = b(\cot.\text{B} - \cot.\text{A})d\text{A}.$$

Hence, in this case for any proposed value of b the error db vanishes when $\text{A} = \text{B}$. Hence the nearer to equality A and B are taken, the less *ceteris paribus* will be the error in b.

In like manner, it may be proved that the nearer to equality A and C are taken, the less will be the error in the computed value of c. Thus we infer in general, that when (as is usually the case in practice) two sides of a triangle are to be computed from the third, the results will be the more exact the nearer the form of the triangle approaches to that of an equilateral triangle.

PROP. LXXXIX.

(288.) *In a right-angled spherical triangle, one oblique angle being given, to determine the relation between the small variations of the sides which include it.*

Let A be the given angle. Hence

$$\cos.\text{A} = \tan.b \cot.c,$$

which being differentiated, gives

$$0 = \cot.c \sec.^2 b db - \tan.b \operatorname{cosec}.^2 c dc,$$

$$\therefore \sin.b \cos.b dc = \sin.c \cos.c db,$$

$$\therefore \sin.2b \,.\, dc = \sin.2c \,.\, db,$$

which is the relation required.

PROP. XC.

(289.) *In a right-angled spherical triangle, the hypothenuse being given, to determine the relation between the small variations of the sides.*

In this case let the equation

$$\cos.c = \cos.a \cos.b$$

be differentiated. This gives

$$\cos.a \sin.b db = -\cos.b \sin.a da,$$

$$\therefore \frac{da}{db} = -\frac{\tan.b}{\tan.a},$$

which is the required relation.

PROP. XCI.

(290.) *Given two sides of a spherical triangle, to determine the relation between the small variations of the angle opposite one of them and the remaining side.*

Let a, b, be the given sides, and A, c, the variable angle and side. By (181), [1]

$$\cos.b \cos.c + \cos.\text{A} \sin.b \sin.c - \cos.a = 0,$$

which being differentiated, gives

$$(-\cos.b \sin.c + \cos.\text{A} \sin.b \cos.c)dc - \sin.\text{A} \sin.b \sin.c d\text{A} = 0,$$

$$\therefore (\cos.\text{A} - \cot.b \tan.c)dc = \sin.\text{A} \tan.c d\text{A},$$

which is the relation sought.

This applies to the method of determining the time by equal altitudes at different sides of the meridian. The tri-

angle is formed by the zenith distance a, the polar distance c, and the zenith distance of the pole b. The variation of the polar distance dc proceeds from the change of declination of the sun between the two observations *.

PROP. XCII.

(291.) *Given two angles of a spherical triangle, to determine the relation between the small variations of the opposite sides.*

Differentiating the equation

$$\sin.a \sin.\text{B} = \sin.b \sin.\text{A}$$

for a and b as variables, we obtain

$$\sin.\text{B} \cos.a da = \sin.\text{A} \cos.b db.$$

Dividing the one by the other, we have

$$\tan.a db = \tan.b da,$$

which expresses the required relation.

(292.) It is unnecessary to extend these examples further. If the student be sufficiently familiar with the necessary principles of the differential calculus, he will find no difficulty in solving any problem of this kind after the examples which have been already given; and if he be not, examples would be unintelligible.

SECTION X.

On geodetical operations.

(293.) One of the most interesting and important applications of trigonomety is the determination of the figure, magnitude, and position of any portion of the earth's sur-

* Woodhouse, Trig. p. 203.

face. *Geodetical* operations have, in general, this for their object. To enter minutely into the principles and details of these operations would not be suitable to the purposes of the present treatise; but it is, nevertheless, necessary to give such a view of them as will enable the student who is desirous of prosecuting the inquiry further, to proceed with greater facility in those less elementary investigations, the difficulties of which he will have to encounter, and as will enable students generally to comprehend the means whereby some of the most striking results of modern practical science have been obtained.

Let us suppose that a survey of a great state, as Great Britain or France, is to be made. In this problem there are three parts perfectly distinct each from the other, and which are solved by means equally distinct from one another. These are

1°. To determine the *figure* of the required tract of country, the configuration of its various internal divisions, and the positions of its cities, towns, &c. relatively to its outlines, so that the whole could be laid down on a map or globe, and might present a *miniature* of the state to be surveyed.

2°. Having thus ascertained its *figure*, to determine its *magnitude*. This will be effected if any one line in it be accurately measured.

3°. To assign the *position* of the country on the surface of the earth relatively to the equator or the pole and the surrounding countries. This may be done by determining the geographical latitude and longitude of any place in the country to be surveyed, the longitudes being supposed to be measured from the meridian of some place relatively to which its position is to be ascertained.

(294.) The end proposed in geodetical operations is not always purely geographical. In those conducted in France

by *Delambre* and *Mechain*, the object was to determine with precision the length of an arc of the meridian extending from *Dunkerque* to *Barcelona*, in order to obtain a permanent standard measure, and the result was the determination of the present French *metre.*

(295.) To simplify our explanation of the method of determining the figure of the tract to be surveyed, we shall, in the first instance, suppose that its surface is perfectly smooth and all at the same level, that is to say, that every point of it is at the same distance from the centre of the earth. Let us suppose that at various points of the country, objects called *signals* are erected, and that they are connected by arcs of great circles. The tract will thus be resolved into a system of spherical triangles, and the sides of these triangles should be of such lengths, that the signals at their extremities may, by the aid of a telescope, be distinctly visible each from the other.

Now by proper instruments placed at each of the signals, let the angle under the sides of the triangle connecting it with two other signals be measured. The methods of measuring these angles, and the instruments whereby the measurement is made, we shall presently explain. This done, the lengths of the sides *in degrees* may be computed, and the spherical polygon, or the system of spherical triangles, which is thus supposed to overspread the country, might be accurately laid down on the surface of a globe. The relative position of all the signals will then be determined. In this first *triangulation*, the signals are supposed to be as distant from one another as is consistent with the distinctness with which it is necessary they should be observed one from another. The triangles of this system are called the *primary triangles.* Each of these triangles is again resolved into a system of smaller triangles by signals instituted within it at convenient places, and similar observations are made

with similar results. These latter triangles are called *secondary triangles*. These will determine the position of the secondary signals with respect to each other and the primary signals.

(296.) Such is the theoretical solution of the problem to determine the figure of a country and its various internal divisions. In practice, however, the problem is attended with greater difficulties, and demands the aid of other principles. It must be considered that the portion of the spherical surface of the earth, which any one, even of the primary triangles occupies, is small compared with the entire surface, and its sides, expressed in parts of a degree, are extremely minute.

The errors of observation which affect the angles of the triangles, though very small, may yet produce upon the sides, expressed in seconds, an effect which may bear a sensible ratio to their whole values. It is therefore desirable that we should have some test by which the correctness of the observations of the angles might be tried, and the error in them, if any, removed or diminished. This end may be attained by a rule established by General Roy, when employed by Government in connecting, by a series of triangles, the Greenwich observatory with the French triangulation extending from the Paris observatory to Calais.

Since all the observed angles belong to spherical triangles, the sum of every three of them in the same triangle must exceed two right angles (142.). The quantity by which it exceeds two right angles is called the *spherical excess.* Let it be ε.

Let the true values of the three angles of any triangle be A, B, C, and the errors of observation α, β, γ; the apparent values will be then

$$A + \alpha, \quad B + \beta, \quad C + \gamma,$$

the errors being negative when the observed angle is less

than the true. The spherical excess computed from the observed angles being ε', we have

$$\text{A} + \text{B} + \text{C} + \alpha + \beta + \gamma - 180^\circ = \varepsilon',$$
$$\therefore \alpha + \beta + \gamma + \varepsilon = \varepsilon',$$
$$\therefore \alpha + \beta + \gamma = \varepsilon' - \varepsilon.$$

Thus it appears that the sum of the errors is the difference between the true and observed spherical excess. If then any means could be found for determining *a priori* the true spherical excess, we should be immediately able to find the sum of the errors accurately. This, however, cannot be done; but we are enabled to find the spherical excess subject to an error by far more minute than any of the errors in the angles, by the ingenious rule which results from the following proposition.

PROP. XCIII.

(297.) *The area of a spherical triangle being known in square feet, it is required to establish a rule for computing the excess of its angles above two right angles in seconds.*

Let ε be the number of seconds in the spherical excess, x number of square feet in the area of the triangle, and r the number of feet in the radius of the sphere. By (262.), if D be the excess related to the radius unity, we have

$$\text{D} = \frac{x}{r^2}.$$

But $\varepsilon = \text{D} \times 206265$, $\therefore$

$$\varepsilon = \frac{x}{r^2} \times 206265,$$

$$l\varepsilon = lx - l\frac{r^2}{206265}.$$

When this rule is applied to the case of a triangle on the surface of the earth, the mean value of the radius r expressed in feet may be easily computed. The number of feet in the

mean length of a degree is 60859.1 × 6. Therefore the number in a second is

$$\frac{(60859.1)\times 6}{60\times 60}=\frac{60859.1}{600},$$

and therefore

$$r=\frac{60859.1}{600}\times 206265$$

in feet.

From this, it follows that

$$l\,\frac{r^2}{206265}=9.3267737,$$

$$\because\ l\varepsilon = lx - 9.3267737.$$

Hence, when the area of a triangle on the surface of the earth is known, the spherical excess may be computed by the following rule:

"*From the logarithm of the number of square feet in the area subtract the constant number* 9.3267737, *and the remainder will be the logarithm of the number of seconds in the spherical excess.*"

(298.) In this method of computing the spherical excess, it is necessary that the area of the triangle should be previously known. As the computation of the area must either immediately or remotely depend on observation or measurement, it must be liable in a greater or less degree to error, and thereby the spherical excess computed by this rule will be also liable to error. If the amount of this error were equal, or nearly equal, to the sum of the errors of the observed angles, the rule would be obviously useless. However, although we are not able to obtain the value of the area with absolute exactness, and thence derive the true value of ε, yet we are able to obtain it within such a degree of accuracy, that the error with which the value of ε, computed by General Roy's rule, will be affected, will not pro-

duce any sensible effect on the value of the sum of the errors of the angles.

(299.) We have already observed, that in order to determine the absolute dimensions of the tract under survey, it is necessary to measure accurately some one line. This line we will suppose a side of the first triangle. Its angles being observed, let the remaining sides and the area be computed, considering the triangle as a plane one. The area thus determined may be used to compute the spherical excess with sufficient accuracy.

If two sides b, c, were known, and the included angle observed, the area would be given by

$$x = \tfrac{1}{2}bc\sin.\text{A},$$

or if the side a and the two angles B and C were known, we should have

$$\sin.\text{A} = \sin.(\text{B} + \text{C}),$$

$$b = a\frac{\sin.\text{B}}{\sin.(\text{B}+\text{C})}, \quad c = a\frac{\sin.\text{C}}{\sin.(\text{B}+\text{C})},$$

$$\because x = \tfrac{1}{2}a^2\frac{\sin.\text{B}\sin.\text{C}}{\sin.(\text{B}+\text{C})};$$

either of these formulæ will give the area with sufficient accuracy to compute the excess. The side a is either the side which has been actually measured, or has been deduced from it by computation.

(300.) If it be desired to determine the extent of the error to which the area is liable, by being computed as a plane area instead of a spherical one, let the formula

$$\tan.\tfrac{1}{2}x = \frac{\tan.\frac{1}{2}a\tan.\frac{1}{2}b\sin.\text{C}}{1+\tan.\frac{1}{2}a\tan.\frac{1}{2}b\cos.\text{C}}$$

be developed by the process of common division, and we obtain a series of the form

$$\tan.\tfrac{1}{2}x = z^2\sin.\text{C} - z^4\sin.\text{C}\cos.\text{C} + z^6\sin.\text{C}\cos.^2\text{C}\cdots\cdot$$

where $z^2 = \tan.\frac{1}{2}a\tan.\frac{1}{2}b$. Now the quantities $\tan.\frac{1}{2}a$, $\tan.\frac{1}{2}b$,

being very small, all the powers of z, after the second, may be neglected, $\therefore$

$$\tan.\tfrac{1}{2}x = \tan.\tfrac{1}{2}a\,\tan.\tfrac{1}{2}b\,\sin.\mathrm{c}.$$

But by developing $\tan.\tfrac{1}{2}x$, we have *

$$\tan.\tfrac{1}{2}x = \frac{x}{2} - \frac{2x^3}{2^3.1.2.3} + \frac{2^4x^5}{2^5.1.2.3.4.5} \cdots$$

and similar series for the other tangents. Making these substitutions, and omitting those terms whose dimensions exceed two, we have

$$\tfrac{1}{2}x = \tfrac{1}{4}ab\sin.\mathrm{c},$$

$$\therefore\ x = \tfrac{1}{2}ab\sin.\mathrm{c}.$$

The quantities neglected are therefore small quantities of the third and higher orders.

(301.) The value of ε being determined by the rule we have just established, let it be subtracted from ε' obtained from observation, and the remainder will be the sum of the errors (296.),

$$\alpha + \beta + \gamma = \varepsilon' - \varepsilon.$$

The errors α, β, γ, are still to be distributed among the angles. If all the three angles A, B, C, have been observed with equal care, and under circumstances equally favourable to accuracy, the sum of the errors should be equally distributed among them, so that

$$\alpha = \frac{\varepsilon' - \varepsilon}{3}, \quad \beta = \frac{\varepsilon' - \varepsilon}{3}, \quad \gamma = \frac{\varepsilon' - \varepsilon}{3}.$$

Thus, in the forty-third triangle of the *Base du Systeme Metrique Decimal*, the three stations are *Lieursaint* (B), *Melun* (C), and *Malvoisine* (A). The side (a), joining the first two, was the base measured. The observed angles were

* Differential Calculus (80.).

$$
\begin{array}{lrl}
 & A' = & 40^\circ . 36' . 56'', 81 \\
 & B' = & 75 \;.\; 39 \;.\; 29,\; 81 \\
 & C' = & 63 \;.\; 43 \;.\; 33,\; 79 \\
\hline
 & & 180^\circ . \;0\; . \;0,\; 41 \\
\varepsilon \text{ computed} \cdots\cdots = & & +\;0,\; 49 \\
\hline
\alpha + \beta + \gamma = & & -\;,\; 08
\end{array}
$$

Hence the correction to be subtracted from each observed angle is

$$\tfrac{1}{3}(\varepsilon' - \varepsilon) = -\,0'',026,$$

neglecting the subsequent places. Hence the true angles are

$$
\begin{aligned}
A &= 40^\circ . 36' . 56'', 836 \\
B &= 75 \;.\; 39 \;.\; 29,\; 836 \\
C &= 63 \;.\; 43 \;.\; 33,\; 816.
\end{aligned}
$$

In this instance the sum of the errors is so small, that it is practically useless to modify the values of the angles by it. It is, however, far from being useless in such a case to compute it, since it serves the important purpose of verifying the accuracy of the observations. Besides, the smallness of its value could not be known until after the computation. We shall presently give an example in which the sum of the errors is more considerable.

In the preceding example the error has been equally distributed amongst the angles; if, however, there be reason to suppose the principal source of error to arise from one of the angles then to this angle the greatest part of the correction should be applied. Or if it be supposed that one of the angles has been taken with extraordinary precision, the correction should be distributed between the other two.

It sometimes happens that one of the angles (A) has not been observed. In that case the spherical excess ε having been computed by the rule, the angle A may be found from

$$\text{A} + \text{B}' + \text{C}' + \beta + \gamma - 180^\circ = s,$$
$$\because \text{A} + \beta + \gamma = 180^\circ + s - (\text{B}' + \text{C}').$$

In which case the computed angle will be affected by the errors of both the angles observed.

In the first triangle of the English survey by General Roy the stations were, Hanger-hill tower, Hampton poor-house, and King's Arbour: and the observed angles were

Hanger-hill tower	42°	2′	32″
Hampton poor-house	67	55	39
King's Arbour	70	1	48
	179	59	59

Here the sum of the observed angles is less than 180° by 1″. This must therefore be a part of the quantity $\alpha + \beta + \gamma$. The computed spherical excess is 0″,15 *. Hence

$$\alpha + \beta + \gamma = 1'',15.$$

This correction is to be distributed among the angles according to the peculiar circumstances attending the observations.

(302.) The angles observed in the first triangle being thus cleared of the errors of observation, and reduced for computation, and also one side called the *base* having been accurately ascertained, suppose in feet, by actual measurement, the next step is to determine the number of seconds in this base considered as an arc of a great circle of the earth's surface. Let x be the number of feet in the base, x'' the number of seconds, D = the number of feet in a degree, and $\text{D}'' = 3600''$ the number of seconds in a degree. We have the proportion

$$x : \text{D} :: x'' : \text{D}'',$$

* Professor Woodhouse has detected an oversight in General Roy's computation. He made the computed excess 0″,29 instead of 0″15.

$$\therefore x'' = \frac{\text{D}''x}{\text{D}} = 3600''\frac{x}{\text{D}},$$

$$\therefore lx'' = l3600 + lx - l\text{D}.$$

If a be the measured base, and that b and c are the sides to be computed, we have

$$\sin.b = \sin.a\frac{\sin.\text{B}}{\sin.\text{A}},$$

$$\sin.c = \sin.a\frac{\sin.\text{C}}{\sin.\text{A}}.$$

(303.) For this computation, it is necessary therefore to have $\sin.a$, and it is convenient that this should be expressed in feet; in which case $\sin.b$ and $\sin.c$ will also be determined in feet. It is therefore requisite to establish formulæ by which a small arc of the great circle being expressed in feet, its sine may be found in feet, and *vice versâ.*

Let x be a small arc in feet, and its sine and cosine expressed in the same units, will be determined by

$$\frac{\sin.x}{r} = \frac{x}{r} - \frac{x^3}{(3)r^3} + \frac{x^5}{(5)r^5} - \frac{x^7}{(7)r^7} + \cdots$$

$$\frac{\cos.x}{r} = 1 - \frac{x^2}{(2)r^2} + \frac{x^4}{(4)r^4} - \cdots$$

where (2), (3) $\cdots$ express

$$(2) = 1.2, \quad (3) = 1.2.3, \quad (n) = 1.2.3 \cdots n,$$

and r is the earth's radius in feet.

Since x is necessarily very small with respect to r, all the terms of these series after the second may be neglected, so that we may assume

$$\sin.x = x\left(1 - \frac{x^2}{6r^2}\right),$$

$$\cos.x = r\left(1 - \frac{x^2}{2r^2}\right).$$

If the latter be expressed relatively to the radius unity, it becomes

$$\cos.x = 1 - \frac{x^2}{2r^2},$$

$$\therefore (\cos.x)^{\frac{1}{3}} = \left(1 - \frac{x^2}{2r^2}\right)^{\frac{1}{3}} = 1 - \frac{x^2}{6r^2},$$

all the higher powers of $\frac{x}{r}$ being neglected as before.

Hence

$$\sin.x = x\sqrt[3]{\cos.x},$$

$$\therefore l\sin.x = lx + \tfrac{1}{3}l\cos.x,$$

where x and $\sin.x$ are expressed in feet. By this formula the arc in feet may be computed from its sine in feet, and *vice, versâ*.

(304.) To illustrate the application of these principles, we shall give the following computations made by Delambre in the French geodetical operations. In this case the arc is supposed to be expressed in *toises* *.

Let

$$x = 14088{,}2858$$
$$\text{D} = 57020.$$

Hence

$$\begin{aligned} lx &= 4{,}14886 \\ l3600 &= 3{,}55630 \\ c.l\text{D} &= 5{,}24397 \\ \hline lx'' &= 2{,}94913, \quad \therefore x'' = 14'\ 49'',5. \end{aligned}$$

To find $\sin.x$ in *toises*,

$$\begin{aligned} lx = l14088{,}2858 \cdots\cdots &= 4{,}14885 \quad 84218 \\ \tfrac{1}{3}l\cos.x = \tfrac{1}{3}l\cos.14'\ 49'',5 \cdots &= 9{,}99999 \quad 86539 \\ \hline l\sin.x \text{ in } toises \cdots\cdots &= 4{,}14885 \quad 70757. \end{aligned}$$

Again, if $\sin.x$ be given, to determine x itself,

* A *toise* is six French feet = 6.396 English feet = 2.132 yards.

$$l\sin.x \text{ in toises} \cdots\cdots = 4{,}14885 \quad 70757$$
$$-\tfrac{1}{3}l\cos.x \cdots\cdots = 9{,}99999 \quad 86539$$
$$lx \cdots\cdots = 4{,}14885 \quad 84218.$$

In the example already given (301.) of the forty-third triangle of the *Base du Système Metrique*, the side a expressed in toises is

$$a = 6075{,}9001.$$

The logarithm of the sine being computed as above, gives

$$l\sin.a = 3{,}78361 \quad 03721.$$

From this and the values of A, B, C, already given, sin.b and sin.c may be computed in toises as follows:

$$c.l\sin.\text{A} = 0{,}18643 \quad 00703$$
$$l\sin.a = 3{,}78361 \quad 03721$$
$$l\sin.\text{B} = 9{,}98625 \quad 01102$$
$$l\sin.b = 3{,}95629 \quad 05526.$$

$$c.l\sin.\text{A} = 0{,}18643 \quad 00703$$
$$l\sin.a = 3{,}78361 \quad 03721$$
$$l\sin.\text{C} = 9{,}95264 \quad 12944$$
$$l\sin.c = 3{,}92268 \quad 17368.$$

The sines of b and c being thus found in *toises*, the arcs themselves may be found by the formula already given, ∴

$$b = 9042{,}5539,$$
$$c = 8369{,}1673.$$

By the process just explained, Delambre calculated the entire system of triangles from Dunkerque to Barcelona in the survey instituted for the measurement of an arc of the meridian.

(305.) The preceding method for resolving a system of triangles proceeds upon the strict rules of spherical trigonometry. This is, however, not the most usual way of proceeding in practice. There are two other methods which

we shall now explain, by which the calculation of the spherical triangles is reduced to that of plane triangles. The first method is founded on a theorem of LEGENDRE, whereby he assigns a plane triangle, the sides of which differ from those of the spherical triangle by a quantity so small as to be neglected without sensible error in the results.

The following is the theorem of Legendre. The demonstration which follows it is taken from a Memoir on Spherical Trigonometry, by LAGRANGE, in the *Journal of the Polytechnic School.*

PROP. XCIV.

(306.) *A spherical triangle, of which the sides* abc *are very small with respect to the radius* r *of the sphere, is equivalent to a plane triangle whose sides are equal to* abc, *and whose angles are* $A - \frac{1}{3}\varepsilon$, $B - \frac{1}{3}\varepsilon$, $C - \frac{1}{3}\varepsilon$, ε *being the excess of the sum of the three angles* A, B, C, *of the proposed spherical triangle above two right angles.*

In the formula

$$\cos.b \cos.c + \cos.A \sin.b \sin.c - \cos.a = 0;$$

a, b, c, are understood to represent the sides in degrees, or to signify the angles subtended by the sides a, b, c, at the centre of the sphere. If a, b, c, be taken to represent the absolute lengths of the sides upon the surface of the sphere, of which the radius is r, it will be necessary to substitute for them in the above formula $\frac{a}{r}$, $\frac{b}{r}$, $\frac{c}{r}$; by which it becomes

$$\cos.\frac{b}{r} \cos.\frac{c}{r} + \cos.A \sin.\frac{b}{r} \sin.\frac{c}{r} - \cos.\frac{a}{r} = 0,$$

the sines and cosines being all related to the radius unity.

For brevity, let

$$\frac{a}{r}=\alpha,\ \frac{b}{r}=\beta,\ \frac{c}{r}=\gamma.$$

By the principles of the Differential Calculus, we have *

$$\sin.\alpha=\frac{\alpha}{1}-\frac{\alpha^3}{1.2.3}+\ldots.$$

$$\cos.\alpha=1-\frac{\alpha^2}{1.2}+\frac{\alpha^4}{1.2.3.4}-\ldots.$$

and we have similar expressions for the sines and cosines of β and γ.

Since the angles α, β, γ, are supposed to be very small, those powers of α, β, γ, whose exponents exceed four, may be neglected, and we shall therefore substitute the first two terms in the series for the sine, and the first three for the cosine. Those substitutions in the last equation will give

$$\left(1-\frac{\beta^2}{1.2}+\frac{\beta^4}{1.2.3.4}\right)\left(1-\frac{\gamma^2}{1.2}+\frac{\gamma^4}{1.2.3.4}\right)$$

$$+\cos.\text{A}\left(\frac{\beta}{1}-\frac{\beta^3}{1.2.3}\right)\left(\frac{\gamma}{1}-\frac{\gamma^3}{1.2.3}\right)-\left(1-\frac{\alpha^2}{1.2}+\frac{\alpha^4}{1.2.3.4}\right)=0.$$

Developing the products, and solving for cos.A, we find

$$\cos.\text{A}=\frac{\beta^2+\gamma^2-\alpha^2}{2\beta\gamma}+\frac{\alpha^4+\beta^4+\gamma^4-2\alpha^2\beta^2-2\beta^2\gamma^2-2\alpha^2\gamma^2}{24\beta\gamma}.$$

Substituting for α, β, γ, their values, this is reduced to

$$\cos.\text{A}=\frac{b^2+c^2-a^2}{2bc}+\frac{a^4+b^4+c^2-2a^2b^2-2b^2c^2-2a^2c^2}{24bcr^2}.$$

Let A′ be the angle opposite to a in the plane triangle, whose sides are a, b, c, ∵

$$\cos.\text{A}'=\frac{b^2+c^2-a^2}{2bc}.$$

Hence

$$\sin.^2\text{A}'=-\frac{a^4+b^4+c^4-2a^2b^2-2b^2c^2-2a^2c^2}{4b^2c^2}.$$

---

* Differential Calculus (73.).

By which the former value of cos.A is reduced to

$$\cos.A = \cos.A' - \frac{bc}{6r^2}\sin.^2A'.$$

Let $A - A' = x$, ∴

$$\cos.A = \cos.(A' + x) = \cos.A'\cos.x - \sin.A'\sin.x.$$

The difference x is so small, that its second and higher powers may be neglected; therefore, by the series for the sine and cosine, we have

$$\sin.x = x, \quad \cos.x = 1,$$

$$\therefore \cos.A = \cos.A' - x\sin.A'.$$

By comparing this with the last value of cos.A, we find

$$x = \frac{bc}{6r^2}\sin.A';$$

and since $A = A' + x$, ∴

$$A = A' + \frac{bc}{6r^2}\sin.A'.$$

The spherical and plane triangles, whose sides are a, b, c, may be considered to be equal in area, since their difference is so very small, that it is unnecessary to take it into account in the present investigation (300.). Let this area be D, ∴

$$D = \tfrac{1}{2}bc\sin.A',$$

$$\therefore A = A' + \frac{D}{3r^2};$$

and for the same reasons,

$$B = B' + \frac{D}{3r^2},$$

$$C = C' + \frac{D}{3r^2},$$

$$\therefore A + B + C = A' + B' + C' + \frac{D}{r^2}.$$

But since A′, B′, C′, are the angles of a plane triangle,

$$A' + B' + C' = \pi,$$

$$\therefore \frac{D}{r^2} = A + B + C - \pi = \varepsilon.$$

The quantity ε is therefore the spherical excess, and we have

$$A' = A - \frac{\varepsilon}{3},$$

$$B' = B - \frac{\varepsilon}{3},$$

$$C' = C - \frac{\varepsilon}{3}.$$

(307.) By this theorem, the spherical triangles which compose any system may be computed by plane trigonometry. The *base* (a) being measured, and the angles of the first triangle observed and reduced for computation, the sides b and c are determined by

$$\sin.b = \sin.a \frac{\sin.B}{\sin.A},$$

$$\sin.c = \sin.a \frac{\sin.C}{\sin.A}.$$

As an example of this method, we shall take one of the French triangles, viz. that found by the stations, *Dammartin* (A), the *Pantheon* (B), and *Saint-Martin* (C). The observed angles, after being reduced by the method (297.), are

A =	57°	52′	53″
B =	46°	22′	30″,2
C =	75°	44′	37″,8
	180°	0	1″
Spherical excess (ε)			1″
	180°	0	0

Hence the angles A′, B′, C′, reduced for computation by Legendre's rule, are

A′ =	57°	52′	52″,67
B′ =	46°	22′	29″,87
C′ =	75°	44′	37″,46
	180°	0	0

To compute b and c, a being known,

$$lb = la + l\sin.\text{B}' + c.l\sin.\text{A}',$$
$$lc = la + l\sin.\text{C}' + c.l\sin.\text{A}',$$

$$\begin{aligned} la &= 4{,}4664126 \\ l\sin.\text{B}' &= 9{,}8596612 \\ cl\sin.\text{A}' &= 0{,}0721433 \\ \hline lb &= 4{,}3982171 \end{aligned}$$

$$\begin{aligned} la &= 4{,}4664126 \\ l\sin.\text{C}' &= 9{,}9864152 \\ c.l\sin.\text{A}' &= 0{,}0721433 \\ \hline lc &= 4{,}5249711 \end{aligned}$$

From whence we find

$$b = 25016, \qquad c = 33494{,}3.\,*$$

(308.) The method of Legendre is not confined to triangles whose sides are very limited in length. Puissant shows that it may be applied to the largest triangle which has ever yet been submitted to calculation, the triangle formed by the stations *Iviza*, *Montgo*, and *Desierto*, being a part of the continuation of the French triangulation into Spain. It may be applied with all the necessary accuracy to cases where the sides exceed a degree and a half. Another singular advantage which it possesses over all other methods is, that it is applicable to triangles described on a spheroid of small eccentricity, and the computations by it are therefore not disturbed by the spheroidal form of the earth.

Thus, this method combines the greatest advantages that any practical rule can possess, expedition and accuracy.

We shall now proceed to explain a third method of cal-

* The sides are expressed in French *metres*. One metre = 39,3702 English inches.

culating geodetical triangles, which has been proposed by Delambre, and by which, as well as by the former two, he computed the whole series of triangles between *Dunkerque* and *Barcelona*.

(309.) This method consists in computing, not the spherical triangles imagined to be found upon the terrestrial sphere, but the plane triangles formed by the chords of the sides of the former. As these chords differ very little from the arcs themselves, a very small reduction will always deduce the one from the other. In order to apply this method, it is necessary therefore to reduce all the spherical angles formed by the sides of the several triangles, and which are obtained and corrected as already explained to the plane angles under the chords of the sides. We shall therefore propose the solution of the following problem, which is of very general utility, and of which the proposed reduction is a particular application.

PROP. XCV.

(310.) *Given two sides* a, b, *of a spherical triangle which differ from quadrants by very small quantities, to investigate formulæ by which the difference between the included angle* C *and the opposite side* c *may be computed when the sides* a, b, *and either the angle* C *or the remaining side* c *are given.*

By (181.)

$$\cos.C = \frac{\cos.c - \cos.a \cos.b}{\sin.a \sin.b}.$$

Let $a + a' = \frac{\pi}{2}$, and $b + b' = \frac{\pi}{2}$, $\because$

$$\cos.C = \frac{\cos.c - \sin.a' \sin.b'}{\cos.a' \cos.b'}.$$

But a', b', being supposed so small, that those powers of them which exceed the second may be neglected, we have *

$$\sin.a' = a', \qquad \sin.b' = b',$$
$$\cos.a' = 1 - \tfrac{1}{2}a'^2, \qquad \cos.b' = 1 - \tfrac{1}{2}b'^2,$$
$$\therefore \cos.\mathrm{c} = \frac{\cos.c - a'b'}{1 - \frac{1}{2}(a'^2 + b'^2)},$$

which by division, omitting as before the higher powers of a', b', gives

$$\cos.\mathrm{c} = \cos.c - a'b' + \tfrac{1}{2}(a'^2 + b'^2)\cos.c,$$
$$\therefore \cos.c - \cos.\mathrm{c} = a'b' - \tfrac{1}{2}(a'^2 + b^2)\cos.c,$$
$$\therefore 2\sin.\tfrac{1}{2}(\mathrm{c} - c)\sin.\tfrac{1}{2}(\mathrm{c} + c) = a'b' - \tfrac{1}{2}(a'^2 + b'^2)\cos.c.$$

As $\sin.\frac{1}{2}(\mathrm{c} - c)$ is necessarily very small, we may without sensible error assume

$$\sin.\tfrac{1}{2}(\mathrm{c} - c) = \tfrac{1}{2}(\mathrm{c} - c) = \tfrac{1}{2}x,$$
$$\sin.\tfrac{1}{2}(\mathrm{c} + c) = \sin.c.$$

Hence we find

$$x = \frac{2a'b' - (a'^2 + b'^2)\cos.c}{2\sin.c}$$
$$= \frac{2a'b'(\cos.^2\frac{1}{2}c + \sin.^2\frac{1}{2}c) - (a'^2 + b'^2)(\cos.^2\frac{1}{2}c - \sin.^2\frac{1}{2}c)}{4\sin.\frac{1}{2}c\cos.\frac{1}{2}c},$$
$$\therefore x = \left(\frac{a'+b'}{2}\right)^2 \tan.\tfrac{1}{2}c - \left(\frac{a'-b'}{2}\right)^2 \cot.\tfrac{1}{2}c.$$

The values of a', b', are here related to the radius as the unit. If they be expressed in seconds, x also expressed in seconds becomes (15.)

$$x = \frac{1}{206265}\left\{\left(\frac{a'+b'}{2}\right)^2 \tan.\tfrac{1}{2}c - \left(\frac{a'-b'}{2}\right)^2 \cot.\tfrac{1}{2}c\right\} \quad [1].$$

which is therefore one of the required formulæ.

If c be given to find $c - \mathrm{c} = y$, the same formulæ will answer the purpose, using c in place of c, so that

$$y = -\frac{1}{206265}\left\{\left(\frac{a'+b'}{2}\right)^2 \tan.\tfrac{1}{2}\mathrm{c} - \left(\frac{a'-b'}{2}\right)^2 \cot.\tfrac{1}{2}\mathrm{c}\right\} \quad [2],$$

---

* Differential Calculus (73.).

because, neglecting all powers of x above the second, we have

$$\tan.\tfrac{1}{2}\mathrm{c} = \tan.\tfrac{1}{2}c.$$

PROP. XCVI.

(311.) *Given two sides and the included angle* (c) *of a spherical triangle described on the earth's surface, to determine the angle under the chords of the sides, the sides being very small with respect to the radius of the sphere.*

Let tangents to the sides through the vertex of the given angle (c) and the chords of the sides be conceived to be produced to the celestial sphere, and let a spherical triangle be formed by great circles through the zenith and the points where the chords produced meet the heavens. The produced tangents will meet the sides of this triangle at the distance $\frac{\pi}{2}$ from the zenith. Let a', b', be the angles under the tangents and the chords, it is evident that the sides of the triangle terminated at the zenith are

$$\frac{\pi}{2} + a', \quad \frac{\pi}{2} + b',$$

and the included angle is c, the third side c being the measure of the angle under the chords. Let $\mathrm{c} - c = x$. By [2] in the last proposition the value of x may be computed. This will be the correction which it is necessary to apply to the horizontal angle (c) in order to reduce it to the angle (c) under the chords.

(312.) The three horizontal angles of each triangle being supposed to be reduced to the angles under their chords, the sum of the reduced angles should be equal to two right angles, and the sum of the corrections should be equal to the spherical excess. If this should not be the case, the

small excess or defect should be distributed among the angles, as explained in (301.).

It will next be necessary to reduce the measured base to its chord. Let the base be b, and its chord c.

$$c = 2\sin.\tfrac{1}{2}b,$$

$$\therefore b - c = \frac{b^3}{24},$$

the higher powers of b being neglected. In this formula b and c are referred to the radius of the sphere as unity. To obtain the correction $b - c$ in feet, let r be the number of feet in the earth's radius, $\because$ (24.)

$$\frac{b-c}{r} = \frac{b^3}{24r^3},$$

$$\therefore b - c = \frac{b^3}{24r^2},$$

which gives the required correction in feet.

(313.) Of the three methods which have been explained for the resolution of geodetical triangles, the first, if the data could be obtained with mathematical exactness, would give the results of the computation also with exactness; and for the same reason, the errors of the data would be involved in the results, though modified by the process of calculation. In matters of calculation, where the data are liable to errors of certain limits, a method of approximation is always as conducive to accuracy as the exact methods, provided that the approximation be within the limits of error in the data. Thus, for example, if the errors in the data were such as to expose a computed angle to an error of $5''$, supposing the method of computation exact; and that, supposing the data exact, the method of approximation would give the computed angle within $0'',1$; it is obvious, that in such case, the result by the approximate method would be practically as accurate as that by the exact method.

For this reason the second method (that of Legendre), though an approximate one, is preferable to the first, on account of the greater simplicity and expedition of the computation.

The third method is exact and expeditious, but it calculates the triangles formed by the chords, and not by the arcs, and a subsequent reduction is necessary if the arcs be required.

(314.) We have hitherto considered that all the triangles were on the surface of an exact sphere, and that their angles and sides were the *immediate* results of observation and calculation. Such, however, is never practically the case. The surface of the earth is covered with asperities and inequalities, small, it is true, and insignificant compared with its whole extent, but very considerable when compared with the sphere of observation from any one point. The objects of observation also are, for obvious reasons, rarely situate upon the mean surface of the terrestrial sphere, but, on the contrary, are conspicuously placed on eminences raised considerably above the level of the sea.

The stations having been selected, some well defined objects are placed above them, which become the points of observation. These should be such as may be distinctly visible, and admit of accurate *bisection* by the observer from adjacent stations. Towers, spires, &c. furnish obvious points of observation, and were frequently used in the geodetical operations carried on in France; but these do not always occur in the most advantageous positions, and the phases which they present, when partially illuminated by the sun, require corrections which are not always certain. The French survey was made with the repeating circle, in which telescopes of very limited power are used; but in this country, where more powerful ones are attached to the in-

struments, a spar of timber planted vertically in a mound of earth for the primary, and a flag-staff for the secondary triangles, are almost exclusively used for day observations.

Other day signals have sometimes been used by French engineers, such as a flat black disc pierced in the centre, the plane of which is placed at right angles to the direction of the observer; the light seen through the aperture in the centre becoming the point of observation.

At night, large fires were formerly used, and subsequently white lights. These latter have, however, been found inconvenient from not continuing to burn for a sufficient length of time. Argand lamps, placed in the foci of parabolic reflectors, have been lately very generally used. But these means will probably be superseded by two, one for day, and one for night observations, which leave nothing to be desired as to power or distinctness. The first is Professor Gaus' method of projecting in any direction a beam of solar light by a mirror connected with an heliostat which, at great distances, appears in the telescope as an intensely lucid point, visible even when the mountain on which it is exhibited is wrapt in mist. The other, for night observations, consists in placing in the focus of a parabolic reflector a small globe of lime, which is intensely heated by flames blown with oxygen gas. The light thus produced is visible at a distance of one hundred miles, and in weather when no other signal would be seen, though less remote *.

A perpendicular from the point of observation to the

* For this beautiful application of a barren fact, we are indebted to Lieut. *Drummond,* of the royal engineers. It has already been used with great effect in the triangulation of Ulster. This talented young officer has also improved the method of *Gaus,* by contriving a very simple apparatus for directing the reflected light instead of the complex heliostat.

horizon, determined by a plumb-line, is called the *axis* of the signal, and is considered as marking the centre of the station.

Any three points of observation being conceived to be connected by straight lines, the plane of the triangle thus formed will be in general inclined at small angles to the horizons of the three stations. If from the vertices of this triangle, or the points of observation, the axes of the signals be supposed to be produced until they meet the surface of the sphere on which those spherical triangles which are the final objects of computation are imagined to be described, the points where they meet it will be the vertices of the spherical triangle, the angles of which, called horizontal angles, are to be determined. If the angles of the plane triangle formed by the points of observation be those which are observed, a reduction or correction is necessary to determine the spherical angles; but the necessity of this reduction depends on the nature of the instrument with which the angles are observed.

The instruments used for angular measurement in geodetical operations usually consist of a circular limb very accurately graduated. They may be divided into two classes, repeating circles and theodolites; one of the distinctions between which is, that the repeating circle measures the angle subtended by two points of observation at a third, while the theodolite measures the horizontal angles. Thus the former requires that its results should be reduced by calculation before they can be applied to the spherical triangles. We shall now describe these instruments more particularly.

The repeating circle.

(315.) The circular limb of this instrument is connected with a vertical pillar, in such a manner, that it has a motion

on an horizontal axis, by which its plane may be inclined at any proposed angle to the plane of the horizon. There are means of clamping it at any required inclination to the horizon. The axis of the vertical pillar which thus supports the limb is placed in the centre of an horizontal circle, which is supported by the tripod which sustains the whole apparatus. The vertical pillar is capable of an azimuth motion, the quantity of which is marked by an index which plays upon the horizontal circle, in the centre of which the axis of the pillar is inserted. Thus it appears that the circular limb has two motions, one round an horizontal, and one round a vertical axis. It is not difficult to perceive that by the combination of these motions, the plane of the limb can always be brought to coincide with the plane of a triangle formed by its centre and any two objects. For by the motion on the horizontal axis, the plane of the limb may be placed so as to be inclined to the horizon at the same angle as the plane of the two objects, and then the circle being clamped so as to maintain that inclination, the azimuth motion on the vertical axis must bring the plane of the instrument to coincide with that of the two objects.

Two telescopes are connected with the circular limb, so that their lines of collimation are parallel to its plane, and placed at opposite sides of it; that which is placed upon the graduated side of the limb is called the superior, and the other the inferior telescope.

Besides the motions round the vertical and horizontal axes already described, the limb is capable of another motion round an axis through its centre and perpendicular to its plane, which therefore is a motion round its centre, and in its own plane.

Each of the telescopes is also capable of moving round the same central axis of the limb, the line of collimation continuing parallel to the plane of the limb.

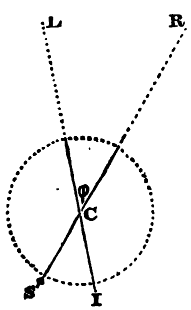
The superior telescope carries with it verniers, and microscopes for *reading off.*

The motion of each telescope round the central axis is independent of the other and of the limb, so that the whole apparatus being fixed, and all the other parts clamped, the observer can move either telescope round the axis of the limb, so as to make it traverse the entire circumference.

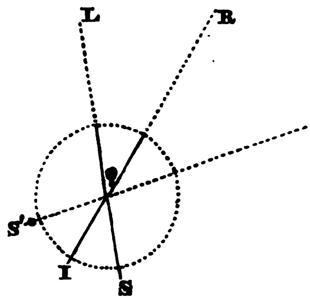
Either telescope may be made to move with the limb round its central axis by clamping it to the limb, or the limb may be made to move round its axis while the telescope remains fixed.

To observe with this instrument the angular distance between two objects R and L, let the plane of the limb be first brought to pass through the objects by moving it on the horizontal and vertical axes. Let the superior telescope be moved round the central axis until its vernier coincides with the zero of the limb, and there let it be clamped to the limb. Let the limb be then turned upon its central axis, carrying the superior telescope with it, until the line of collimation of the superior telescope shall be directed to the point of observation R.

This is done by bringing the intersection of two fine wires placed in the focus of the eyeglass accurately upon the object R. The limb being then fixed in its position, let the inferior telescope be brought upon the object L, and then clamped to the limb.



Let S and I be the eyeglasses of the superior and inferior telescopes, directed as already described to the objects R and L. Let the limb be now turned round its centre C in its own plane, and on its central axis, carrying the telescopes with it until the inferior telescope I shall be directed to R. By this change I is moved to where S was, and S is



moved back to S', so that S'I is the present position of the instrument is the same arc as SI in the former.

This done, the superior telescope being relieved of the clamp, is moved towards L, the limb remaining fixed, until it is brought upon the object L, and then it is clamped again. It is evident, that if the divisions were exact, and the observations accurately made, the vernier carried by S would, by this process, be moved from zero on the limb to that point whose distance from zero is double the angle subtended by the objects R and L at the centre of the instrument.

By a repetition of this process, the superior telescope may be again moved forward upon the limb through another arc equal to double the observed angular distance of the objects, and that after n repetitions of the process, the angle being read off at the vernier carried by S, the value will be $2n\phi$. In case the number of repetitions be such that the telescope S shall have gone once or oftener round the entire circumference, let m be the number of circuits, and α the angle read off. To determine ϕ, the angle under the objects, we shall have

$$2n\phi = 2m\pi + \alpha,$$

$$\therefore \phi = \frac{2m\pi + \alpha}{2n} = \frac{m}{n}\pi + \frac{\alpha}{2n}.$$

The advantage of this beautiful principle of repetition is, that it spreads any error of division or inexactitude in reading off, over so great a number of observations as to produce no sensible effect upon the observed angle. If the efficacy of this principle be admitted to its full extent, it ' '"

go so far as to render a small instrument susceptible of as high a degree of accuracy as the largest and most costly astronomical circles. It is only necessary to have the observations taken with great care, and multiplied until the repetition will give the desired degree of accuracy.

Besides the method of observing by *crossed observations*, which has been just described, there is another method of taking angles with the repeating circle, which it is proper to mention. Having, as before, fixed the superior telescope with its vernier at zero, and directed to the object R, let the inferior telescope be moved upon the limb until it also bisects the object R, and in that position let it be clamped upon the limb. The superior telescope being then disengaged, is moved upon the limb until the object L is bisected by its wires. The vernier will then have moved over an arc of the instrument equal to the angle under the objects, and the angle may be immediately read off if greater accuracy be not required. But if repetition be necessary, the superior telescope being clamped upon the limb, the limb is turned on its axis until the superior telescope is again made to bisect the object R, and then the inferior telescope, being brought as before also to bisect R, is clamped upon the limb, and the superior telescope again made to bisect the object L. The vernier will give double the angular distance, and the same process being continued, any number of repetitions may be made, and any degree of accuracy obtained as before. When many repetitions are to be made, this process is not so expeditious as the former. The inferior telescope is only used in this case to ascertain whether the limb has changed its position while the superior telescope has been moved.

The theodolite.

(316.) The limb of the theodolite, like that of the re-

peating circle, is a circle graduated with great accuracy. This circle is permanently connected by a number of strong conical radii or spokes to a central pillar, the axis of which is perpendicular to the plane of the limb. This central pillar being hollow, is placed upon a solid vertical pillar, which is made to fit it, and on which it turns with a perfectly smooth motion. When properly adjusted, the axis on which the instrument turns is truly vertical, and the plane of the limb is truly horizontal, and continues so while the instrument turns upon its axis. The central hollow pillar to which the limb is attached, rising to some distance above its plane, terminates in an apparatus which supports a telescope, whose line of collimation is in a vertical plane through the axis of the instrument.

The supports of this telescope are such as to permit its line of collimation to move in the vertical plane, so as to be directed to any proposed altitude. A vertical semicircle properly graduated is attached to it, which determines the altitude of the object to which it points. The line of collimation of this telescope is therefore capable of two motions, one in azimuth, of which the limb of the instrument partakes, and one in altitude, during which the limb is stationary.

The limb of the instrument thus moveable is surrounded by a fixed frame, which is provided with means of supporting several microscopes over the upper surface of the limb, and by which the angles are read off. These microscopes are placed with their axes vertical immediately over fixed verniers under which the graduated limb moves.

In the fixed frame, and beneath the plane of the limb, is placed a telescope, whose line of collimation is in a vertical plane through the axis of the instrument, and is level, or nearly so.

The instrument being supposed to be adjusted, and its

axis brought accurately to coincide with the axis of the station, let us suppose that the horizontal angle under two other stations be required. Let the inferior telescope be made to bisect any distant object, and let the superior or transit telescope be made to bisect the signal at one of the proposed stations L. If the vertical plane through the line of collimation be supposed to pass through zero of the limb, and the verniers be severally observed, they will show the angular distances of the signal L from each of them; let these distances be

$$D', D'', D''' \dots$$

Let the transit telescope be now moved, together with the limb upon the vertical axis, until the signal of the other proposed station R is bisected. Then having examined the inferior telescope, to ascertain whether the frame carrying the verniers has remained fixed, which is proved by the same distant object continuing to be bisected, let the verniers be again observed, and let the angles read off be

$$d', d'', d''' \dots$$

It is evident that between the two observations an arch of the limb equal to the difference of the azimuths of the two signals, or what is the same, the horizontal angle under them, must have passed under each vernier.

Hence the horizontal angles observed at the verniers severally are

$$D' - d', D'' - d'', D''' - d''' \dots$$

the mean of which is to be taken as the value of the observed angle.

Reichenbach, the celebrated German artist, has added the repeating principle to the theodolite.

In cases where the horizontal angle only is required, the theodolite has the advantage of the repeating circle, inasmuch as it gives the horizontal angle immediately without any subsequent reduction; whereas the repeating circle

taking the actual angle under two signals, a correction is necessary to reduce this to the horizon.

(317.) We now propose to investigate the correction necessary for reducing the angular distance between two objects observed with the repeating circle to the horizontal angle.

Let the altitudes of the objects, which are always small, be a', b', and the angular distance observed be c. In the spherical triangle formed by the zenith and the two objects projected on the heavens, the two sides terminated at the zenith are nearly quadrants, and the angle C at the zenith is the sought horizontal angle, being the difference of the azimuths of the two objects. The quantity $\mathrm{C} - c = x$ is the correction which it is necessary to apply to c to find C; hence by [1], (310.),

$$x = \frac{1}{206265}\left\{\left(\frac{a'+b'}{2}\right)^2 \tan.\tfrac{1}{2}c - \left(\frac{a'-b'}{2}\right)^2 \cot.\tfrac{1}{2}c\right\},$$

which gives the required correction.

(318.) To facilitate this reduction in practice, tables have been computed, which give the value of x when $(a' + b')$, $a' - b'$, and c are known. Such tables may always be computed from the formula just established for a succession of values of these quantities $a' + b$, $a' - b'$, and c, which are called the *arguments* of the tables.

The zenith distances of the objects, which give the altitudes a', b', should be observed at the same time, or nearly so, with the angle c, in order that no change in the quantity of refraction may affect the accuracy of the reduced angle.

(319.) It has been already observed, that there is in the axis of each signal a certain visible point distinctly marked, which forms the point of observation from other stations. If, in observing the angles of the plane triangle formed by three signals, the centre of the repeating circle could be brought accurately to coincide with that point in [illegible] of

each signal which forms the point of observation from the other signals, the three observed angles would, if added, amount to 180°. This, however, is not the case; for in general the centre of the instrument cannot be brought to coincide with the point of observation in the axis of each signal; and it frequently happens that the centre of the limb cannot even be brought into the axis of the signal. From these causes it happens that the three angles which are the results of immediate observation are not in the same plane, and if they were, they would be frequently not the angles of the same plane triangle. Hence it arises that their sum would not be accurately equal to 180°, even if the observations were taken with the utmost exactness.

Two corrections are therefore necessary to reduce the observed angles to those of a plane triangle joining three points in the axes of the three signals; one for reducing the observed angles to the plane of these points, and the other for reducing them to the axes of the signals. The former correction is unnecessary when the angles are taken with a theodolite, since they are immediately horizontal angles without any reduction; but the latter correction is necessary when the centre of the theodolite is not exactly in the axis of the signal.

(320.) We shall first suppose that at the three stations the centre of the instrument has been placed accurately in the axes of the signals, and that the angles are only to be reduced to the same plane. The centre of the instrument being supposed in the axis of the station A, let right lines be supposed to be drawn from it to the points of observation of the signals B and C. Also from the same point let lines be drawn to these points of the axes of the signals B and C at which the centre of the instrument had been placed in taking the angles at these stations. The angle under the former lines is the observed angle, and that under the latter the re-

duced angle. The investigation of the required correction, therefore, resolves itself into the following problem.

PROP. XCVII.

(321.) *An angle observed in a plane making a small angle with the horizon is projected upon another plane also making a small angle with the horizon, and passing through the vertex of the observed angle by vertical circles through the sides of the observed angle and the station of the observer, it is required to investigate the correction which it is necessary to apply to the observed angle to reduce it to its projection.*

Let the arcs of the verticals through the sides of the observed angle between them and the horizon, or, what is the same, the altitudes of the observed objects be a and b, and let the altitudes of their projections on the required plane be a' and b'. Let the observed and reduced angles be α and α'. The corrections which would reduce these angles to the horizon are (317.),

$$x = \frac{1}{r''}\left\{\left(\frac{a+b}{2}\right)^2 \tan.\tfrac{1}{2}\alpha - \left(\frac{a-b}{2}\right)^2 \cot.\tfrac{1}{2}\alpha\right\},$$

$$x' = \frac{1}{r''}\left\{\left(\frac{a'+b'}{2}\right)^2 \cot.\tfrac{1}{2}\alpha' - \left(\frac{a'-b'}{2}\right)^2 \cot.\tfrac{1}{2}\alpha'\right\},$$

where $r'' = 206265$. It is obvious that the correction which reduces α to α' is $x - x'$. Since $\alpha - \alpha'$ is very small, we may assume $\tan.\frac{1}{2}\alpha = \tan.\frac{1}{2}\alpha'$. Let

$$\delta a = a - a', \quad \delta b = b - b',$$

$$\therefore \left(\frac{a+b}{2}\right)^2 - \left(\frac{a'+b'}{2}\right)^2 = \tfrac{1}{2}(\delta a + \delta b)[(a + b) - \tfrac{1}{2}(\delta a + \delta b)],$$

$$\left(\frac{a-b}{2}\right)^2 - \left(\frac{a'-b'}{2}\right)^2 = \tfrac{1}{2}(\delta a - \delta b)[(a - b) - \tfrac{1}{2}(\delta a - \delta b)].$$

Hence we obtain

$$x - x' = \frac{1}{2r''}\left\{(\delta a + \delta b)[(a + b) - \tfrac{1}{2}(\delta a + \delta b)]\tan.\tfrac{1}{2}\alpha \right.$$
$$\left. - (\delta a - \delta b)[(a - b) - \tfrac{1}{2}(\delta a - \delta b)]\cot.\tfrac{1}{2}\alpha\right\}.$$

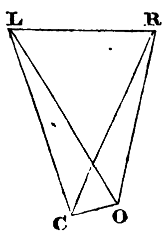
(322.) This formula will serve to reduce the observed angles to the same plane, the quantities δa, δb, being estimated by the distances between the place of the centre of the instrument and the points of observation at each of the stations.

The same formula will serve for computing the effect of refraction on the observed angle by substituting for δa and $\delta a'$ the known values of the refractions.

(323.) Whether the repeating circle or theodolite be used, it may, and frequently does happen, that the centre of the limb cannot be brought into the axis of the signal. The observed angle must therefore be corrected, and a formula determined by which this reduction may be made.

PROP. XCVIII.

(324.) *The angle under two distant objects being observed, it is required to investigate a formula by which the angle under which the same objects would be observed from a near point in the same plane may be computed.*



Let O be the centre of the instrument with which the observation is made, L and R the objects, and C the centre of the station to which the angle O is to be reduced, and LOR (which we shall call O) the observed angle.

Let the distances CL and CR be l and r, and let the distance CO and the angle COL be accurately measured, and let CO $= a$ and COL $= y$. Now, since in the triangles LCR and LOR the three angles in each

are equal to the same sum 180°, and by the change of the vertex from O to C, CLO is added to, and CRO subtracted from, the angles L and R, it follows that

$$LCR = O + CRO - CLO.$$

Let LCR $= x$, $\because$

$$x - O = CRO - CLO.$$

But

$$\sin.CRO = \frac{a}{r}\sin.COR = \frac{a}{r}\sin.(O + y),$$

$$\sin.CLO = \frac{a}{r}\sin.y.$$

Since the angles CRO and CLO are very small, their sines may be substituted for them without sensible error. Hence

$$x - O = a\left(\frac{\sin.(O+y)}{r} - \frac{\sin.y}{l}\right);$$

or if this be expressed in seconds,

$$x - O = \frac{a}{\sin.1''}\left(\frac{\sin.(O+y)}{r} - \frac{\sin.y}{l}\right)^*.$$

In the application of this formula it is necessary that the distances r and l should be known, but great accuracy in their values is not requisite. They will be determined with sufficient exactness by resolving the triangle LOR as a plane triangle, assuming the angles observed as the true values of the angles L, O, R.

The above formula may be considered inconvenient from its having two terms. A formula with only one may be easily derived from it.

We have

---

* In retaining this formula in the memory, some technical assistance may be derived from remembering that r is the distance of the right-hand object, and l of the left. Also, y is the angle under the centre of the station C, and the left-hand object L.

$$l = \frac{r\sin.\text{B}}{\sin.(\text{B}+x)},$$

and since $x = o$ nearly,

$$l = \frac{r\sin.\text{B}}{\sin.(\text{B}+o)}.$$

Hence

$$x - o = \frac{a}{r\sin.1''}\left\{\frac{\sin.(o+y)\sin.\text{B} - \sin.(\text{B}+o)\sin.y}{\sin.\text{B}}\right\},$$

which being developed, gives

$$x - o = \frac{a\sin.o\,\sin.(\text{B}-y)}{r\sin.1''},$$

to which logarithms are immediately applicable.

(325.) Having now explained the principal corrections which are necessary in order to reduce the angles which are the immediate results of observation to the angles of the spherical triangles which are imagined to be described on the mean surface of the earth supposed spherical, it remains to explain shortly how this mean surface is to be determined. There is a certain level which the waters of the ocean would always maintain if undisturbed by the action of the sun and moon in the production of tides. LAPLACE has proved that this is a mean between the highest and lowest state to which their surface is reduced by the attractions of these bodies. With this constant level a sphere is imagined to be described, and on the surface of this sphere all geodetical triangles and all distances are conceived to be projected.

It is necessary, therefore, to be able to reduce all distances measured upon any part of the earth's surface, such as the base of a system of geodetical triangles, to this standard level. For this purpose let the radius of the spherical surface which has the mean level of the sea be r, and the height of the surface of a measured base above this level be h. The base B may then be considered as an arc of a circle having the radius $r + h$, and is to be reduced to

a similar arc having the radius r. Let b be the reduced base.

$$\frac{B}{b}=\frac{r+h}{r},$$

$$\therefore B-b=\frac{Bh}{r+h}=\frac{Bh}{r}\left(1+\frac{h}{r}\right)^{-1},$$

developing this last by common division, we find

$$B-b=B\left(\frac{h}{r}-\frac{h^2}{r^2}+\frac{h^3}{r^3}\cdots\right).$$

Since h is always very small with respect to r, it is never necessary to take more than two terms of this series.

The value of h may be determined with great precision; but the details of the process could not be suitably introduced into this treatise.

(326.) We have observed that it is necessary that some one line or base which is generally a side of the first triangle of the series should be accurately measured, that from it, by the aid of the observed angles, all the other sides of the triangles may be computed. Since then the accuracy of the whole process depends on the correctness of this measurement, it should be executed with extraordinary care and precision. The ground selected should be as level and free from inequalities as possible, and the base should be as long as the nature of the ground permits. Every part of the base should be in the same vertical plane, so that, setting apart small inequalities of the surface, it may be an arc of a great circle of the earth. If straight rods of a substance which would not be liable to a change of magnitude with a change of temperature could be obtained, and which would admit of being placed end to end in close contact, the length of the base could be found with the greatest precision. But this is not the case; all substances are liable to change their dimensions with their temperature, and it becomes necessary to ascertain the temperature of whatever measures be used

at the time they are applied to determine the base, and to determine the change of length corresponding to every change of temperature.

Rods of various substances have been used in the measurement of the bases in the British and French surveys. In the former, rods of deal and of glass were used by General Roy in 1784, and steel chains by Col. Mudge in 1791. The degree of exactness attained in these operations may be conceived from the fact that the two measurements of a base on Hounslow Heath of upwards of 27400 feet in length differed by less than three inches. In the French survey rods of platinum were used.

(327.) In order to check the observations, measurements, and computations, it is usual to measure one or more sides of the triangles which are remote from the base from which they are all computed. By a comparison of the values of these obtained from computation and from measurement, the accuracy of the results may be determined. The lines measured for this purpose are called *bases of verification.*

In the survey conducted by General Roy, the original base was on Hounslow Heath, extending from King's Arbour to Hampton poor-house, a distance of 27404 feet, and a base of verification was measured on Romney Marsh, near Dover, the length of which was 28533 feet. In this case the length of the Romney base computed differed from the length measured by only twenty-eight inches.

(328.) We have hitherto considered the figure of the earth, setting aside small inequalities, as a perfect sphere, and it may be regarded as such in most geodetical operations. Its true figure is, however, an oblate spheroid * of small

* An oblate spheroid is a solid generated by the revolution of an ellipse round its lesser axis.

eccentricity, the axis of which is the polar diameter. The meridians, which are sections of the surface by planes through the axis, are therefore equal ellipses, the eccentricity of which is that of the spheroid itself. The transverse axes of all the meridians are in the plane of the equator, and the centre of the equator is their common centre. The parallels of latitude are obviously all circles.

As in some calculations it has been found necessary to take into account the spheroidal form of the earth, we shall here determine a few of the formulæ most necessary in such investigations.

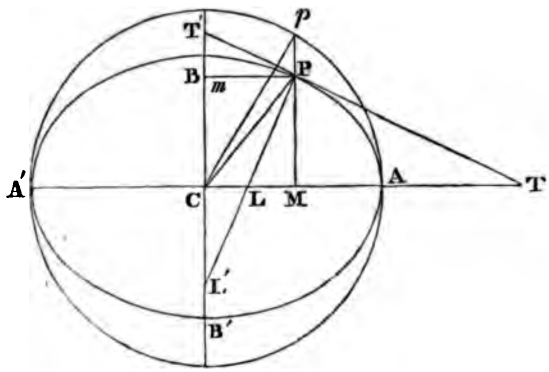
The horizon of any place being the tangent plane to the spheroid at that place, the vertical line, or that which points to the zenith, must be the normal to the meridian drawn through the place. If the latitude be defined to be the angle under this normal and the plane of the equator, it is easy to show that the principle that "the latitude of the place equals the altitude of the celestial pole" is true of the spheroid.

(329.) We now propose to establish formulæ expressing the various quantities connected with the position of a place on the terrestrial spheroid in terms of its polar and equatoreal diameters and the latitude of the place.

Let ABA′B′ be a meridian of the earth, CB $= b$ being the polar, and CA $= a$ the equatorial semidiameter. Let a circle be described upon AA′ as diameter. Let P be any place upon the meridian, and let the normal and tangent PLL′ and TPT′ be drawn to intersect both axes. Let PM, Pm, be perpendiculars to the axes from P. Let PM be produced to meet the circumscribed circle at p, and let Cp and CP be drawn.

Let $\lambda =$ the angle PLT, which is the latitude of the place.

$\omega =$ the angle PCT, or the geocentric latitude.



$\lambda' =$ the angle pCT, called the reduced latitude.

N = PL, T = PT, $x =$ Pm = CM.

N$'$ = PL$'$, T$'$ = PT$'$, $y =$ PM, $y' = p$M.

$z =$ CP, $\theta =$ CPL, $s =$ PA.

$$e = \frac{\sqrt{a^2 - b^2}}{a}.$$

(330.) The equation of the ellipse referred to the axes AA$'$, BB$'$, is *

$$a^2y^2 + b^2x^2 = a^2b^2.$$

The angle under the normal and the transverse axis is determined by

$$\tan.\lambda = \frac{a^2y}{b^2x},$$

but

$$\frac{y}{x} = \tan.\omega,$$

$$\therefore \tan.\omega = \frac{b^2}{a^2}\tan.\lambda.$$

Hence we find

* See Geometry (128.), (187.)

$$\cos.^2\omega = \frac{a^4\cos.^2\lambda}{a^4\cos.^2\lambda + b^4\sin.^2\lambda} = \frac{\cos.^2\lambda}{1 - 2e^2\sin.^2\lambda + e^4\sin.^2\lambda},$$

$$\sin.^2\omega = \frac{b^4\sin.^2\lambda}{a^4\cos.^2\lambda + b^4\sin.^2\lambda} = \frac{(1-e^2)^2\sin.^2\lambda}{1 - 2e^2\sin.^2\lambda + e^4\sin.^2\lambda}.$$

(331.) To determine λ, we have *

$$\frac{\tan.\lambda'}{\tan.\omega} = \frac{y'}{y} = \frac{a}{b},$$

$$\therefore \tan.\lambda' = \frac{a}{b}\tan.\omega = \frac{b}{a}\tan.\lambda,$$

$$\therefore \sin.^2\lambda' = \frac{b^2\sin.^2\lambda}{a^2\cos.^2\lambda + b^2\sin.^2\lambda} = \frac{(1-e^2)\sin.^2\lambda}{1 - e^2\sin.^2\lambda},$$

$$\cos.^2\lambda' = \frac{a^2\cos.^2\lambda}{a^2\cos.^2\lambda + b^2\sin.^2\lambda} = \frac{\cos.^2\lambda}{1 - e^2\sin.^2\lambda}.$$

Let $\frac{1}{1 - e^2\sin.^2\lambda} = M^2$, $\therefore$

$$\sin.\lambda' = (1 - e^2)^{\frac{1}{2}}M\sin.\lambda, \quad \cos.\lambda' = M\cos.\lambda,$$

(332.) To determine θ,

$$\tan.\theta = \tan.(\lambda - \omega) = \frac{\tan.\lambda - \tan.\omega}{1 + \tan.\lambda\,\tan.\omega},$$

$$\therefore \tan.\theta = \frac{e^2\tan.\lambda}{1 + (1-e^2)\tan.^2\lambda} = \frac{e^2\sin.\lambda\cos.\lambda}{1 - e^2\sin.^2\lambda},$$

$$\therefore \tan.\theta = M^2e^2\sin.\lambda\cos.\lambda.$$

(333.) To determine z †,

$$z^2 = \frac{b^2}{1 - e^2\cos.^2\omega} = \frac{a^2(1-e^2)}{1 - e^2\cos.^2\omega},$$

$$\therefore z^2 = \frac{a^2(1 - 2e^2\sin.^2\lambda + e^4\sin.^2\lambda)}{1 - e^2\sin.^2\lambda},$$

$$z^2 = M^2a^2(1 - 2e^2\sin.^2\lambda + e^4\sin.^2\lambda),$$

$$= a^2(1 - e^2\sin.^2\lambda').$$

(334.) To determine x and y, we have

$$x = z\cos.\omega, \qquad y = z\sin.\omega,$$

* Geometry (195.) † Geometry (173.)

$$\therefore x = \text{M}a\cos.\lambda, \quad y = \text{M}a(1 - e^2)^{\frac{1}{2}}\sin.\lambda.$$

(335.) To determine N and N′, we have *,

$$\text{N}^2 = \frac{b^2}{a^2}(a^2 - e^2x^2) = \frac{b^2}{a^2}(a^2 - \text{M}^2a^2e^2\cos.^2\lambda)$$

$$= b^2(1 - \text{M}^2e^2\cos.^2\lambda) = \text{M}^2a^2(1 - e^2)^2,$$

$$\therefore \text{N} = \text{M}a(1 - e^2),$$

$$\text{N}'^2 = \frac{a^2}{b^2}(b^2 + \frac{a^2e^2}{b^2}y^2) = a^2 + a^2e^2\text{M}^2\sin.^2\lambda,$$

$$\therefore \text{N}' = \text{M}a.$$

(336.) To determine T and T′,

$$\text{T} = \text{N}\tan.\lambda = \text{M}a(1 - e^2)^{\frac{1}{2}}\tan.\lambda,$$

$$\text{T}' = \text{N}'\cot.\lambda = \text{M}a\cot.\lambda.$$

(337.) To determine the radius of curvature, we have †

$$\text{R} = \frac{b'^3}{ab},$$

b' being the semidiameter conjugate to z, ∴

$$bb' = a\text{N}, \quad \therefore \text{R} = \frac{a^2}{b^4}\text{N}^3,$$

$$\therefore \text{R} = \frac{\text{M}^3a^5(1-e^2)^3}{b^4} = \text{M}^3a(1 - e^2).$$

(338.) To determine ds, we have ‡

$$ds = \text{R}d\lambda = \text{M}^3a(1 - e^2)d\lambda.$$

(339.) The factor M, which enters all the preceding expressions, can be reduced to a form in which it is capable of being developed in a series of cosines or sines of the multiples of the latitude λ. We have

$$\text{M}^2 = \frac{1}{1-e^2\sin.^2\lambda} = \frac{2}{2-2e^2\sin.^2\lambda},$$

$$\therefore \text{M}^2 = \frac{2}{2-e^2+e^2\cos.2\lambda}.$$

---

* Geometry (191.) † Geometry (387.)

‡ Differential Calculus (140.)

Let $n = \frac{e^2}{2-e^2}$, ∵

$$\text{M}^2 = \frac{1+n}{1+n\cos.2\lambda}.$$

Under this form M, or its powers, or lM, can be reduced to a converging series of the form

$$\text{A} + \text{B}\cos.2\lambda + \text{C}\cos.4\lambda + \text{D}\cos.6\lambda + \cdots$$

The details of the process for reducing it to this series could not properly be introduced here.

(340.) By the preceding formula all the lines of the terrestrial spheroid corresponding to a given latitude may be computed when a and e are known. These are determined by a comparison of two meridional arcs measured in different latitudes.

Let s be the actual length of an arc of the meridian, and let x be the difference of the latitudes of its extremities, we have

$$s = \text{F}(x);$$

and by Maclaurin's series *,

$$s = \text{A}_1\frac{x}{(1)} + \text{A}_2\frac{x^2}{(2)} + \text{A}_3\frac{x^3}{(3)} + \text{A}_4\frac{x^4}{(4)} + \cdots$$

where $\text{A}_1, \text{A}_2, \cdots$ signify what $\frac{ds}{dx}$ and its differential coefficients become when $x = 0$, and

$$(2) = 1.2, \quad (3) = 1.2.3, \quad (4) = 1.2.3.4, \text{ \&c.}$$

By (338.),

$$\frac{ds}{dx} = \text{R} = \text{M}^3 a(1 - e^2).$$

The value of M being substituted, and the successive differential coefficients found, we obtain the series

$$s = \text{R}x + \tfrac{3}{2}\text{M}^2\text{R}e^2\left\{\sin.2\lambda\frac{x^2}{(2)} + 2\cos.2\lambda\frac{x^3}{(3)} - 2^2\sin.2\lambda\frac{x^4}{(4)} - \cdots\right\}$$

* Differential Calculus (57.).

$$s = \text{R}x + \tfrac{3}{8}\text{M}^2\text{R}e^2\left\{\sin.2\lambda\left(\frac{y^2}{(2)} - \frac{y^4}{(4)} + \frac{y^6}{(6)} - \cdots\right)\right.$$
$$\left. + \cos.2\lambda\left(\frac{y^3}{(3)} - \frac{y^5}{(5)} + \frac{y^7}{(7)} \cdots\cdots\right)\right\}$$

where $y = 2x$. But

$$y - \sin.y = \frac{y^3}{(3)} - \frac{y^5}{(5)} + \frac{y^7}{(7)} - \cdots$$

$$1 - \cos.y = \frac{y^2}{(2)} - \frac{y^4}{(4)} + \frac{y^6}{(6)} - \cdots$$

Hence, neglecting e^4, &c. we have

$$s = \text{R}x + \frac{3}{8}\text{R}e^2\left\{\sin.2\lambda(1 - \cos.2x) + \cos.2\lambda(2x - \sin.2x)\right\}$$

$$= \text{R}x + \frac{3}{4}\text{R}e^2\left\{x\cos.2\lambda - \sin.x\cos.(2\lambda + x)\right\}$$

$$= \text{R}x\left\{1 + \frac{3}{4}e^2\cos.2\lambda - \frac{3}{4}e^2\cos.(2\lambda + x)\frac{\sin.x}{x}\right\}.$$

But

$$\sin.x = x(1 - \frac{x^2}{(3)} \cdots),$$

$$\cos.^{\frac{1}{3}}x = (1 - \frac{x^2}{(2)} \cdots)^{\frac{1}{3}} = 1 - \frac{x^2}{(3)} \cdots$$

Hence we obtain

$$s = \text{R}\left\{1 + \frac{3}{4}e^2\cos.2\lambda - \frac{3}{4}e^2\cos.^{\frac{1}{3}}x\cos.(2\lambda + x)\right\}.$$

In this formula x is understood to be expressed in reference to the radius unity. If it be expressed in seconds, as is usually the case, the formula becomes

$$s = \frac{x}{180^0}\text{R}\pi\left\{1 + \frac{3}{4}e^2\cos.2\lambda - \frac{3}{4}e^2\cos.^{\frac{1}{3}}x\cos.(2\lambda + x)\right\}.$$

(341.) From this formula we may deduce one expressing an arc of one degree, more simply than the former, and sufficiently accurate. Let $x = 1$, and $e = \sin.\text{I}$, $\because$

$$\text{R} = \text{M}^3 a\cos.^2\text{I},$$

$$\therefore s = \frac{\text{M}^3 a\pi}{180} \cos.^2\text{I} \left\{ 1 + \frac{3}{4} \sin.^2\text{I} (\cos.2\lambda - \cos.^{\frac{1}{2}} 1^\circ \cos.(2\lambda + 1^\circ) \right\},$$

for which, without sensible error, we may take

$$s = \frac{\text{M}^3 a\pi}{180} \cos.^2\text{I} \left\{ 1 + \frac{3}{4} \frac{\pi}{180} \tan.^2\text{I} \sin.2\lambda \right\}.$$

To determine the eccentricity of the terrestrial spheroid, let degrees be measured in different latitudes, and let s and s' be their absolute lengths. The values of s and s' being expressed by the preceding formula in terms of the latitudes, let a be eliminated, and the value of I will be determined by

$$\sin.^2\text{I} = \frac{1 - m^{\frac{2}{3}}}{\sin.^2\lambda - m^{\frac{2}{3}} \sin.^2\lambda' - \frac{\pi}{360}\left(\sin.2\lambda' - m^{\frac{2}{3}} \sin.2\lambda\right)},$$

where $m = \frac{s'}{s}$, and λ, λ', are the latitudes of the southern extremities of s and s'. By this formula, sin.I being determined, we have

$$e = \sin.\text{I},$$

$$\frac{b}{a} = \sqrt{1 - e^2} = \cos.\text{I}.$$

(342.) Having explained the general principles on which geodetial operations are conducted, it may be useful to give an example of their actual application. For this purpose we shall take the operations carried on by Delambre and Mechain by order of the French government to obtain a standard for the decimal system of measures. In order that the standard should be permanent and subject to no accidental changes, it was proposed that the length of a quadrant of a meridian should be accurately determined, and this being divided into 10,000,000 equal parts, one of these parts should be adopted as the unit of length, and that all

measures of length should be referred to it, and should be either multiples or submultiples of it. This unit is called a *metre* *. Ten metres, a hundred metres, a thousand metres, ten thousand metres, &c. are called decametre, hectometre, kylometre, myriometre, &c.; and the tenth, hundredth, thousandth parts of a metre are called decimetre, centimetre, millimetre, &c. the multiples being named by Greek, and the submultiples by Latin derivatives.

(343.) In order to obtain the length of the quadrant it was determined to measure an arc of the meridian extending from the parallel of Dunkerque to that of Barcelona, an extent of about ten degrees of latitude. A chain of triangles was accordingly formed between these points along the meridian. This triangulation for about two degrees from Dunkerque to Paris, is represented in the plate, p. 218. The angles and sides of the primary triangles are marked by the larger characters, and those of the secondary triangles by the smaller ones. We shall suppose the angles and sides of the primary triangles to be measured and computed by some or all of the methods already explained. The question then is, by the secondary triangles formed by the sides of the primary triangles (produced if necessary) and the meridian, to compute the several parts into which the meridian is divided at the different points where the sides of the system of primary triangles intersect it.

The latitude of the point D (Dunkerque) and the azimuth of W (Watten) from D being obtained from observation, the latitude of W, and the azimuth of D from W, and the difference of longitudes of D and W, may be computed. For (the earth being supposed spherical), a spherical triangle will be formed by the pole (P) and the points D and W. In

---

* For the length of a metre relatively to our measures, see p. 183, note.

this triangle PD, the colatitude of D, and the angle PDW, the azimuth of W from D (or its supplement, according as the azimuths are measured from the north or south), are obtained by observation. DW is a side of one of the primary triangles, and has been therefore previously computed. The angle at P is the difference of longitudes of D and W. The data therefore in this triangle are two sides PD, DW, and the included angle PDW, and the quantities sought are the difference of the sides PD and PW, the angle P and the difference of the angles D and W. These being all very small quantities, the ordinary formulæ may be modified, so as to give ease and expedition to the process of computation, and all the necessary accuracy to the results.

By continuing this process through the whole series of triangles, the latitudes and longitudes of the several stations may be computed, also the respective azimuth of each station from the horizons of the others, and the perpendicular distances of the several stations from the meridian and from the perpendicular to it through the first station D.

(344.) The first of the series of primary triangles formed by the stations Dunkerque, Watten, and Cassel (C), is divided into two triangles by the meridian D*a*. In the triangle DW*a*, the side DW and the angle W are known from primary system of triangles; the azimuth angle WD*a* is known from observation. Hence by the usual formulæ D*a*, W*a*, and the angle *a* may be computed.

Also WC being known from the primary system, we have

$$\text{WC} - \text{W}a = \text{C}a,$$

which is therefore obtained. Also the angle *b*C*a* is known from the primary triangle WCF (*Fiefs*). Hence in the triangle *b*C*a*, C*a*, and the angles *a* and C are given, to compute *ab*, C*b*, and the angle *b*.

By continuing this process, we shall successively solve the triangles *b*F*d*, *d*F*e*, *sef*, &c. &c. This process being

carried through the whole chain of triangles, the length of the entire arc from Dunkerque to Barcelona was obtained by adding the several parts D*a*, *ab*, *bd*, *de*, &c.

As a verification of these results, not only the entire arc, but also the subdivisions of it, were computed in another way, and by principles and data different from, and independent of, those used in the investigation just explained.

In the triangle DC*b*, the azimuth CD*b* being obtained by observation, and the side DC and the angle DC*b* being known from the primary triangles, the sides D*b*, C*b*, and the angle *b*, may be computed. This value of D*b* serves as a verification of that obtained by adding D*a* and *ab* computed by the triangles DW*a* and *a*C*b*. Again, in the triangle *b*F*e*, the side *b*F is known by CF − C*b*, and the angle F is known from the primary triangles; hence *be*, F*e*, and the angle *e* are computed, and *be* serves as a verification of the value of the same quantity obtained by adding *bd* and *de* derived from the triangles *b*F*d* and *d*F*e*. And by continuing this process through the whole chain of triangles, every part of the one computation was checked and verified by the corresponding parts of the other.

It cannot but be interesting to examine a few of the results of these two computations, the comparison of which shows the extreme skill and precision with which they must have been conducted. In the following table will be found the computation of the arc of the meridian from Dunkerque to the eighth intersection (*i*) deduced from each series of secondary triangles. The successive points of verification *b*, *e*, &c. are called *nodes*, being the points where the two series of triangles intersect. The distances are all expressed in *toises*.

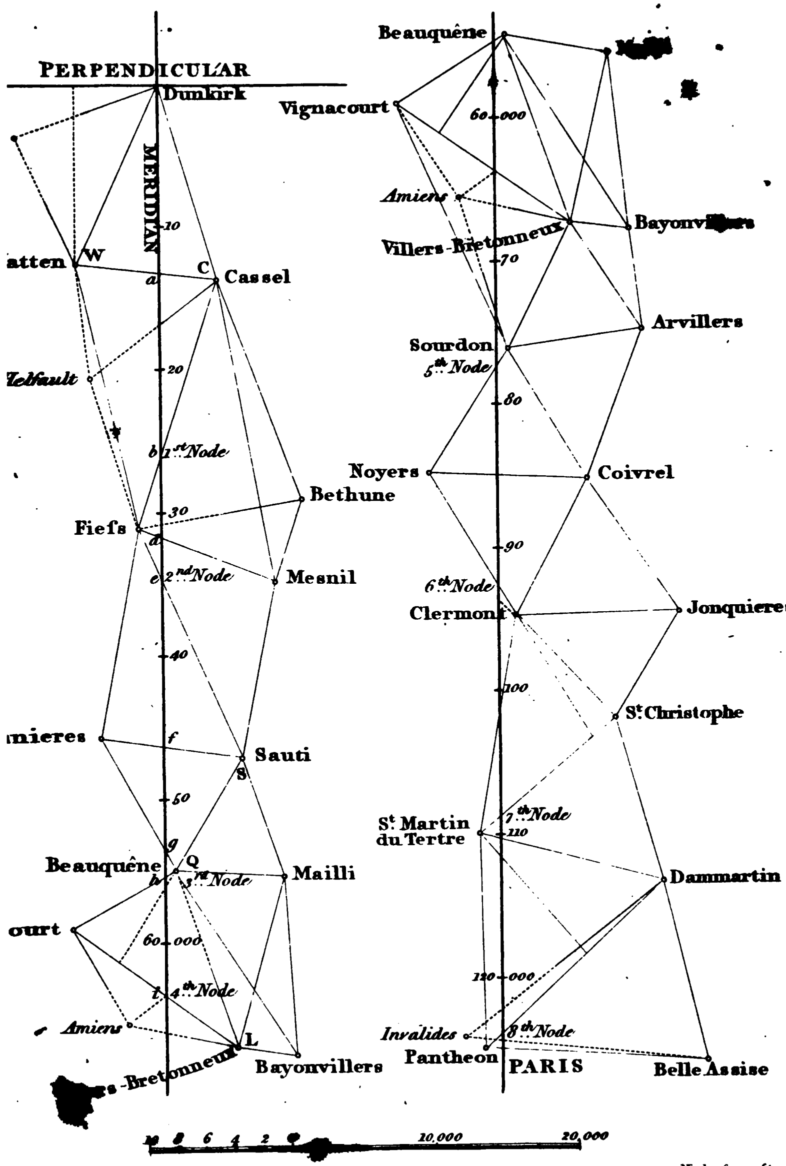
First series of secondary triangles.

No.	Angles.	Angles.	Sines of angles.	Sines of opp. sides.	Partial and total arcs. Log. of partial arcs.
1	aDW	25°19′42″.14	9.63124.637	3.75411.245	Da= 12785.981
	aWD	74 28 45.28	9.98386.686	4.10673.294	log.Da=4.10673.405
	WaD	80 11 33.27	9.99360.631	4.11647.240	
	Dunkerque.	0.69			
2	cab	80 11 33.27	9.93360.631	4.07515.321	ab= 11875.247
	acb	79 48 35.35	9.99309.481	4.07464.170	log.ab=4.07464.265
	abc	19 59 51.85	9.53400.453	3.61555.143	Db= 24661.228
	Cassel.	0.47			
3	Fbd	19 59 51.85	9.53400.453	3.35345.178	log.bd=3.81935.406
	bFd	91 11 19.40	9.99990.652	3.81935.376	bd= 6597.115
	bdF	68 48 48.88	9.96960.668	3.78905.392	Db= 24661.228
	Fiefs.	0.13			Dd= 31258.343
4	dFe	42 59 49.63	9.83375.991	3.54823.350	de= 3533.732
	Fde	111 11 11.12	9.96960.668	3.68408.026	log.de=3.54823.358
	deF	25 48 59.32	9.63897.819	3.35345.178	
	Fiefs.	0.07			De= 34792.075
5	Sef	25 48 59.33	9.63897.819	3.76803.561	log.ef=4.04110.259
	eSf	54 45 8.66	9.91204.436	4.04110.178	ef= 10992.655
	efS	99 25 52.62	9.99408.953	4.12314.695	De= 34792.075
	Sauti.	0.61			Df= 45784.730
6	Bfg	99 25 52.62	9.99408.953	3.95805.399	fg= 7247.415
	fBg	51 56 49.41	9.89621.832	3.86018.277	log.fg=3.86018.313
	Bgf	28 37 18.27	9.68035.817	3.64432.262	
	Bonnieres.	0.30			Dg= 53032.145
7	Qga	28 37 18.27	9.68035.817	3.03584.634	ga= 2027.905
	aQg	116 33 40.58	9.95155.931	3.30704.748	log.ga=3.30704.751
	Qag	34 49 1.15	9.75660.339	3.11209.156	
	Beauquène.	0.00			Da= 55060.049
8	Qgh	28 37 18.27	9.68035.817	2.85722.654	log.gh=3.17662.955
	gQh	91 54 0.07	9.99976.116	3.17662.954	gh= 501.860
	ghQ	59 28 41.66	9.93522.319	3.11209.156	Dg= 53032.145
	Beauquène.	0.00			Dh= 54534.005
9	Vhi	59 28 41.66	9.93522.319	3.90530.848	hi= 8476.654
	hVi	65 14 50.05	9.95814.466	3.92822.396	log.hi=3.92822.445
	hiV	55 16 28.83	9.91481.493	3.88489.423	
	Vignacourt.	0.54			Di= 63010.659

es of secondary triangles for verification.

		Angles.	Sines of angles.	Sines of opp. sides.	Total and partial arcs. Log. of the partial arcs.
	D*cb*	143° 13′ 41″.52	9.77715.806	4.39201.060	D*b* = 24661.229
1	*c*D*b*	16 46 27.59	9.46030.065	4.07515.319	log.D*b* = 4.39201.471
	D*bc*	19 59 51.85	9.53400.453	4.14885.708	
	Cassel.	0.96			
	F*bc*	19 59 51.85	9.53400.453	3.68408.023	log.*bc* = 4.00564.573
2	*b*F*c*	134 11 9.03	9.85556.934	4.00564.504	*bc* = 10130.846
	*bc*F	25 48 59.53	9.63897.824	3.78905.393	D*b* = 24661.229
	Fiefs.	0.21			D*c* = 34792.075
	S*ca*	25 48 59.53	9.63897.824	4.00752.319	log.*ca* = 4.30681.399
3	*c*S*a*	119 22 0.65	9.94026.625	4.30681.120	*ca* = 20268.097
	S*ac*	34 49 1.15	9.75660.339	4.12314.735	D*c* = 34792.075
	Sauti.	1.3			D*a* = 55060.173
	VQ*b*	24 39 40.51	9.62039.904	3.54424.270	
4	QV*b*	65 14 50.05	9.95814.466	3.88198.833	
	V*b*Q	90 5 29.70	9.99999.948	3.92384.314	
	Vignacourt.	0.26			
	bai	34 49 1.15	9.75660.339	3.65699.705	*ai* = 7950.484
5	*abi*	89 54 30.30	9.99999.948	3.90039.313	log.*ai* = 3.90039.356
	bia	55 16 28.83	9.91181.495	3.81520.859	D*a* = 55060.173
		0.28			D*i* = 63010.657

A Part of the Triangulation conducted by M.M. Delambre and Mechain for the purpose of Measuring an arc of the meridian between the parallels of Dunkirk and Barcelona.



Neeles Sc. 352 Stra

By comparing the several results of these two computations, we find,

by the first series,	by the second series,
$Db = 24661.228$,	$Db = 24661.229$,
$De = 34792.075$,	$De = 34792.075$,
$Da = 55060.049$,	$Da = 55060.173$,
$Di = 63010.659$.	$Di = 63010.657$.

Difference of the values of $Db = 0.001$,
. . . . $De = 0.000$,
. . . . $Da = 0.264$,
. . . . $Di = 0.002$.

Hence it appears that, except in the third case, the difference of the results of the two computations do not exceed the 500th of a toise; and even in the third case, the difference does not amount to three-tenths of a *toise*. The entire arc Di is in length about 134338 English yards, and the two computations agree in its value to within a small fraction of an inch.

The arc of the meridian between the parallels of Dunkerque and Barcelona computed by the whole of the first series of secondary triangles is in *toises*

551583.765 = 1175986.587 English yards,

and by series for verification,

551583.512 = 1175986.048 English yards.

The difference of these results is less than nineteen inches, the entire arc being above a million of English yards in length.

The length of the whole arc of the meridian, and also those of its several parts being thus ascertained, it is necessary to determine the differences of the latitudes of the extremities of the several partial arcs, in order to obtain the angles under the normals through their several extremities, and, by comparing these with the lengths of the several arcs,

to compute the eccentricity of the meridian. The differences of latitudes of the several points may be obtained by observing the zenith distances of the same star when on the meridian at both extremities of the arc, the difference of the two zenith distances will be the angle under the normals or the difference of the latitudes, if the star be at the same side of the zenith at both places, and the sum if it be at opposite sides *. If the latitude of any one station be known, the latitudes of the others may thus be found. In this manner the latitudes of Dunkerque, Paris, Evaux, Carcasonne, and Montjouy, were obtained. By comparing the results with those obtained by a measurement made in Peru in 1736, it was concluded that the excess of the equatoreal diameter above the polar $= \frac{1}{334}$.

That part of the process which consists in determining the difference of the latitudes of the extremities of each partial arc, is not susceptible of the same degree of accuracy as that which determines the actual length of the arc. The observation of the difference of latitudes is generally liable to an error of about two seconds, no accidental cause being supposed to disturb the plumb line. Now a second of latitude answers to about 33 yards of the meridian. Besides this, it is well known that the attractions of mountains, &c. frequently displace the plumb line by several seconds.

We have already noticed the great degree of accuracy of which the terrestrial measurement is susceptible by a comparison of the results of the two series of secondary triangles. The computation was, however, submitted to another equally severe test. A base of verification upwards of seven miles long was measured at upwards of four hundred miles distant

---

* The observed zenith distances are to be corrected for *refraction*, *aberration*, &c.

from the first base; and upon comparing the length obtained from absolute admeasurement with the length obtained from the computation of the chain of triangles, the two results did not differ by one foot.

The instrument used in this survey was the repeating circle.

SECTION XI.

On the computation of trigonometrical tables.

(345.) It has been already observed, that trigonometrical computation is conducted by the aid of computed tables from which the numerical values of the sines, tangents, &c. of angles referred to some given radius may be found. Of these tables there are two kinds; those in which the immediate values of the sines, &c. are registered, and which are called tables of *natural sines*, &c. and those in which the logarithms of the sines, &c. are registered, and which are called tables of logarithmic sines, &c. It is not proposed here to enter minutely into the details of the methods of constructing trigonometrical tables, but only to point out in a general way the application of the formulæ by which the successive terms of a table may be computed, and the methods of checking the errors, whether of the computist or the printer, in those tables which have been already computed.

(346.) It very rarely happens that the values of the sines or cosines of angles can be exactly expressed by integers or finite decimals. An approximation in decimals can, however, be always obtained to any degree of accuracy which may be required. The ordinary trigonometrical tables give

the values continued to seven places of decimals; but tables have been computed extending to ten, and even to fifteen, decimal places. If the sines of angles, which are nearly equal to 90°, or the cosines of very small angles be required to that degree of approximation which would determine the results to seconds and tenths, it will be necessary to extend the computation to twelve decimal places, the radius being unity.

(347.) The radius, with respect to which the ordinary tables are computed, is 100000. The values of the sines, cosines, &c. obtained from such tables may be converted into the corresponding values related to unity, or *vice versâ*, by moving the decimal point five places to the left or to the right (24.). We shall, as usual, suppose them related to unity as radius.

(348.) If we suppose that the series of angles of which the sines and cosines are to be tabulated, are in arithmetical progression the common difference, and the number of places in the approximate values will depend each upon the other. If A be any angle of the table, and x the common difference, then several successive angles of the table will be

$$\begin{array}{l} A \\ A + x, \\ A + 2x, \\ A + 3x. \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \end{array}$$

Now if upon calculation it be found that the sines of several of these successive angles agree in the first seven places, it is plain that seven places do not give a sufficient approximation to distinguish angles differing by so small a quantity as x. Let us suppose that the computed value of the sines of A, A + x and A + $2x$, were the same as far as

seven places, but that the seventh place in sin.$(A + 3x)$ were different; it is obvious that in this case it would be useless to tabulate sin.$(A + x)$, sin.$(A + 2x)$, and that if the approximation be limited to seven places, that sin.$(A + 3x)$ should succeed sin.A, and therefore that the common difference should be $3x$; or if the common difference x be retained, the approximation must be continued until the last digit of sin.$(A + x)$ differ from that of sin.A. It should also be observed, that the degree of approximation necessary to distinguish angles, having a given difference x, also depends on the values of the angles themselves, since the variation of the sine is very slow with respect to the variation of the arc, if it be nearly 90°, and that of the cosine if it be very small. In calculations requiring an extreme degree of accuracy, therefore, it is frequently necessary to compute the values of the sines or cosines to a greater number of places than are given in the tables.

(349.) If x be the least angle in the proposed table, and the successive terms be the multiples of x, $2x$, $3x$, $4x$, &c. any three successive tabulated angles will be

$$(n - 1)x,\ nx,\ (n + 1)x,$$

and by (43.) [4], we have the relation between their sines and cosines;

$$\sin.(n + 1)x = \sin.(n - 1)x + 2\sin.x \cos.nx,$$
$$\cos.(n + 1)x = \cos.(n - 1)x - 2\sin.x \sin.nx.$$

By these formulæ, if the first term sin.x be known, and also the sines of $(n - 1)x$ and nx, the sine of $(n + 1)x$ may be computed. That is, if the first term of the tables, and any two successive terms be known, all the succeeding ones may be computed. By substituting successively 1, 2, 3, · · · · for n, we obtain

$$\left.\begin{aligned}\sin.2x &= 2\sin.x \cos.x \\ \cos.2x &= 1 - 2\sin.^2 x\end{aligned}\right\}$$
$$\left.\begin{aligned}\sin.3x &= \sin.x + 2\sin.x \cos.2x \\ \cos.3x &= \cos.x - 2\sin.x \sin.2x\end{aligned}\right\}$$

$$\left.\begin{aligned}\sin.4x &= \sin.2x + 2\sin.x\cos.3x\\ \cos.4x &= \cos.2x - 2\sin.x\sin.3x\end{aligned}\right\}$$

. . . .

. . . .

by which, if the first term of each series be known, all the others may be found.

(350.) To determine the first terms, we have *

$$\sin.x = \frac{x}{1} - \frac{x^3}{(3)} + \frac{x^5}{(5)} - \frac{x^7}{(7)} + \cdots\cdot$$

$$\cos.x = 1 - \frac{x^2}{(2)} + \frac{x^4}{(4)} - \frac{x^6}{(6)} + \cdots\cdot$$

As x is always small, these series converge sufficiently to determine $\sin.x$, $\cos.x$, with considerable facility. Thus, if x be one degree $= 3600''$, we have

$$x = \frac{3600}{206265} = \frac{240}{13751},$$

$$\because \sin.x = \frac{240}{13751} - \frac{(240)^3}{(13751)^3 \times 6} + \frac{(240)^5}{(13751)^5 \times 120} - \cdots\cdot$$

$$\cos.x = 1 - \frac{(240)^2}{(13751)^2 \times 2} + \frac{(240)^4}{(13751)^4 \times 24} - \cdots\cdot$$

If x be one minute, we have

$$x = \frac{60}{206265} = \frac{4}{13751},$$

in which case the convergence is still more rapid.

In order that the values expressed in a given number of decimal places should be the nearest possible to the true value, the computation should be continued to a few additional places. Thus, suppose the sine of $1'$ were computed only to six decimal places, we should find its value (A),

$$\text{A} = ,000290;$$

---

* Differential Calculus (73.)

but if the computation were continued to three additional places, we should find its value (B),

$$B = ,000290888.$$

Hence it follows that its value in six places is more nearly expressed by (C),

$$C = ,000291$$

than (A). For if we find the excess of C above the more approximate value B, we shall find it much less than the excess of the latter above A.

$$\begin{aligned} C &= ,000291 \\ B &= ,000290888 \\ \hline \therefore C - B &= ,000000112 \\ B &= ,000290888 \\ A &= ,000290 \\ \hline \therefore B - A &= ,000000888 \end{aligned}$$

the one difference being nearly eight times the other. In all calculations in which approximate decimal values are used, this remark should be attended to.

(351.) The sines and cosines being found, the tangents, cotangents, secants, and cosecants, may be determined by

$$\tan.x = \frac{\sin.x}{\cos.x}, \quad \cot.x = \frac{\cos.x}{\sin.x},$$

$$\sec.x = \frac{1}{\cos.x}, \quad \operatorname{cosec}.x = \frac{1}{\sin.x}.$$

(352.) By the form of the series for the $\sin.x$, it is easy to perceive that the sines of very small angles are in the same ratio as the angles themselves, for

$$\frac{\sin.x}{x} = 1 - \frac{x^2}{(3)} + \frac{x^4}{(5)} \cdots$$

which is subject to imperceptible variation when x is very small: or what will, perhaps, be considered more satisfactory, its variation, whatever it be, will not appear within the number of decimal places to which the tables are computed.

Upon this principle the sines of angles less than one minute may be computed. Let n be a number less than 60. We have

$$\sin.n'' : \sin.1' :: n : 60,$$

$$\therefore \sin.n'' = \sin.1' \times \frac{n}{60} = ,000291 \times \frac{n}{60},$$

$$\therefore \sin.n'' = 00000485 \times n;$$

or if still greater accuracy be desired, take

$$\sin.1' = ,000290888204,$$

$$\therefore \frac{\sin.1'}{60} = ,0000048481367,$$

$$\therefore \sin.n'' = ,0000048481367 \times n.$$

(353.) When the common difference of the tables is 1', the process of calculation may be greatly expedited by taking the successive orders of differences until one be found, which, for the requisite number of places, is constant. Then the terms of the tables need only be computed at certain intervals, the intermediate ones being filled up by values derived from the constant differences *.

When the table has been computed up to 30°, the succeeding terms may be obtained by a more expeditious process. By the formulæ

$$\sin.(60^\circ - x) = \cos.(30^\circ - x) - \sin.x,$$
$$\cos.(60^\circ - x) = \cos.x - \sin.(30^\circ - x).$$

By which the sines and cosines of the arcs from 30° to 60° may be computed, those of the arcs less than 30° being previously known.

By the principles,

* Differential and Integral Calculus (508.), *et seq.* This is the principle upon which Mr. Babbage's ingenious machinery for calculating tables is constructed. The machine now in process of construction extends to six orders of differences.

$$\sin.(60^\circ + x) = \cos.(30^\circ - x),$$
$$\cos.(60^\circ + x) = \sin.(30^\circ - x),$$

the cosines and sines of the arcs from 60° to 90° are the same as the sines and cosines of those below 30°.

In like manner in the computation of the tangents and cotangents, those of the arcs from 0° to 45° are the same as the cotangents and tangents from 45° to 90°.

(354.) In order to check the errors of the tables, certain formulæ have been selected, called *formulæ of verification.* The student will find no difficulty in establishing the following, which are taken from Professor Woodhouse's Trigonometry.

$$\sin.30^\circ = \frac{1}{2}, \quad \sin.45^\circ = \frac{1}{\sqrt{2}},$$
$$\sin.60^\circ = \frac{\sqrt{3}}{2}, \quad \sin.18^\circ = \frac{\sqrt{5}-1}{4}.$$
$$\sin.9^\circ = \cos.81^\circ = \tfrac{1}{4}[(\sqrt{5}+3)^{\frac{1}{2}} - (5-\sqrt{5})^{\frac{1}{2}}],$$
$$\cos.9^\circ = \sin.81^\circ = \tfrac{1}{4}[(\sqrt{5}+3)^{\frac{1}{2}} + (5-\sqrt{5})^{\frac{1}{2}}],$$
$$\sin.27^\circ = \cos.63^\circ = \tfrac{1}{4}[(5+\sqrt{5})^{\frac{1}{2}} - (3-\sqrt{5})^{\frac{1}{2}}],$$
$$\cos.27^\circ = \sin.63^\circ = \tfrac{1}{4}[(5+\sqrt{5})^{\frac{1}{2}} + (3-\sqrt{5})^{\frac{1}{2}}],$$

and the following formula of Euler,

$$\sin.x + \sin.(36^\circ - x) + \sin.(72^\circ + x) = \sin.(36^\circ + x) + \sin.(72^\circ - x).$$

By substituting for x in these any proposed value, the values of the sines, &c. resulting from the tables should render the equation identical, if they be correct.

PART III.

THE ANALYSIS OF ANGULAR SECTIONS.

PART III.

THE ANALYSIS OF ANGULAR SECTIONS.

SECTION I.

Of multiple arcs.

(355.) The formulæ which exhibit the relations between two arcs and their sum and differences may be easily transformed into others which determine the relations between the sines, cosines, &c. of successive multiples of the same arc. If in the formulæ established in Part I. Sect. III. we substitute $n\omega$ and ω for ω and ω', we obtain the following results:

$$\sin.(n \pm 1)\omega = \sin.n\omega \cos.\omega \pm \sin.\omega \cos.n\omega,$$
$$\cos.(n \pm 1)\omega = \cos.n\omega \cos.\omega \mp \sin.n\omega \sin.\omega,$$
$$\sin.(n + 1)\omega + \sin.(n - 1)\omega = 2\sin.n\omega \cos.\omega,$$
$$\sin.(n + 1)\omega - \sin.(n - 1)\omega = 2\sin.\omega \cos.n\omega,$$
$$\cos.(n + 1)\omega + \cos.(n - 1)\omega = 2\cos.n\omega \cos.\omega,$$
$$\cos.(n + 1)\omega - \cos.(n - 1)\omega = - 2\sin.n\omega \sin.\omega,$$
$$\sin.(n + 1)\omega \sin.(n - 1)\omega = \sin.^2 n\omega - \sin.^2\omega,$$
$$\cos.(n + 1) \cos.(n - 1)\omega = \cos.^2 n\omega - \sin.^2\omega,$$
$$\frac{\sin.(n+1)\omega}{\sin.(n-1)\omega} = \frac{\tan.n\omega + \tan.\omega}{\tan.n\omega - \tan.\omega},$$
$$\frac{\cos.(n+1)\omega}{\cos.(n-1)\omega} = \frac{1 - \tan.n\omega \tan.\omega}{1 + \tan.n\omega \tan.\omega},$$

$$\tan.(n \pm 1)\omega = \frac{\tan.n\omega \pm \tan.\omega}{1 \mp \tan.n\omega\tan.\omega}.$$

(356.) The whole theory of angular sections, and all the various relations between the trigonometrical functions of an angle and its multiples, may be derived from two remarkable formulæ established by EULER, by which the sine and cosine of an angle are expressed as *exponential functions* of the angle itself. These formulæ we propose, in the first instance, to establish, and from them to derive all the theorems which form the subject of this part. They will thus hold the same relation to the analysis of angular sections as the formula for the sine of the sum of two angles does to Plane Trigonometry, and as the formulæ established in (181.) do to Spherical Trigonometry. Were it not that the principles upon which these exponential formulæ are established are not sufficiently elementary, the original formula for the sine of the sum of two angles itself might be deduced from them, and these celebrated theorems might thus be made the foundation of the whole superstructure of trigonometry.

It is true, that by adopting these theorems as the foundations of all the investigations through which we are about to conduct the reader, we shall, in some few instances, derive our formulæ from considerations less elementary than might otherwise be necessary; but the clear and systematical connexion which the different parts of the subject will thus acquire, and the singleness and uniformity of the reasoning by which we shall be conducted through them, will much more than compensate for the absence of one or two insulated elementary proofs, which, were they introduced, would not at all harmonize with the other parts of the investigation.

(357.) The exponential formulæ for the sine and cosine which we are now to establish are

$$\sin.x = \frac{1}{2\sqrt{-1}}\left(e^{x\sqrt{-1}} - e^{-x\sqrt{-1}}\right),$$

$$\cos.x = \frac{1}{2}\left(e^{x\sqrt{-1}} + e^{-x\sqrt{-1}}\right).$$

We shall give two methods of investigating these theorems.

First method *.

(358.) Let

$$y = \sin.x, \qquad z = \cos.x,$$
$$\therefore dy = \cos.xdx, \quad dz = -\sin.xdx,$$
$$\therefore dy = zdx, \qquad dz = -ydx;$$

multiplying the first by $\sqrt{-1}$, and adding the results, we obtain

$$dz + \sqrt{-1}dy = (z + \sqrt{-1}y)dx\sqrt{-1},$$
$$\therefore \frac{d(z+\sqrt{-1}y)}{z+\sqrt{-1}y} = \sqrt{-1}\,dx.$$

The first member of this equation is the differential of the hyperbolic logarithm of $z + \sqrt{-1}.y$, and $\therefore$

$$l(z + \sqrt{-1}y) = \sqrt{-1}.x,$$
$$\therefore \cos.x + \sqrt{-1}.\sin.x = e^{x\sqrt{-1}},$$

no constant is added, since both sides become equal when $x = 0$ †.

* It will be necessary before the student can proceed with this part of trigonometry, that he should acquire a knowledge of the first principles of the Differential Calculus, scil. the differentiation of algebraic, exponential, logarithmic, and trigonometrical functions, and Taylor's theorem. It would be also desirable that he should know the fundamental principles of the Integral Calculus. The first six sections of my work on the Differential Calculus, and the first section of the Integral Calculus, will probably be found sufficient.

† Differential and Integral Calculus (21.), (191.)

If the sign of x be changed, this becomes

$$\cos.x - \sqrt{-1}\sin.x = e^{-x\sqrt{-1}},$$

which being added to and subtracted from the former, gives

$$\left.\begin{aligned} \cos.x &= \tfrac{1}{2}\left(e^{x\sqrt{-1}} + e^{-x\sqrt{-1}}\right) \\ \sqrt{-1}\sin.x &= \tfrac{1}{2}\left(e^{+x\sqrt{-1}} - e^{-x\sqrt{-1}}\right) \end{aligned}\right\} \cdots\cdots [1].$$

Second method.

(359.) The developments of $\cos.x$ and $\sin.x$, given by Taylor's or Maclaurin's theorem, are

$$\cos.x = 1 - \frac{x^2}{(2)^*} + \frac{x^4}{(4)} - \frac{x^6}{(6)} + \cdots\cdots$$

$$\sin.x = \frac{x}{(1)} - \frac{x^3}{(3)} + \frac{x^5}{(5)} - \cdots\cdots$$

The developments of $e^{x\sqrt{-1}}$ and $e^{-x\sqrt{-1}}$ by the same means are

$$e^{x\sqrt{-1}} = \left(1 - \frac{x^2}{(2)} + \frac{x^4}{(4)} - \cdots\cdots\right),$$

$$+\sqrt{-1}\left(\frac{x}{(1)} - \frac{x^3}{(3)} + \frac{x^5}{(5)} \cdots\cdots\right).$$

$$e^{-x\sqrt{-1}} = \left(1 - \frac{x^2}{(2)} + \frac{x^4}{(4)} - \cdots\cdots\right),$$

$$-\sqrt{-1}\left(\frac{x}{(1)} - \frac{x^3}{(3)} + \frac{x^5}{(5)} - \cdots\cdots\right).$$

Eliminating the series within the parentheses by the developments of $\sin.x$ and $\cos.x$, we obtain

$$\left.\begin{aligned} e^{x\sqrt{-1}} &= \cos.x + \sqrt{-1}\sin.x \\ e^{-x\sqrt{-1}} &= \cos.x - \sqrt{-1}\sin.x \end{aligned}\right\} \cdots\cdots [2].$$

---

* See p. 209.

From which, by addition and subtraction, we get

$$\cos.x = \frac{1}{2}\left(e^{x\sqrt{-1}} + e^{-x\sqrt{-1}}\right),$$

$$\sin.x = \frac{1}{2\sqrt{-1}}\left(e^{x\sqrt{-1}} - e^{-x\sqrt{-1}}\right).$$

The former of these investigations is given by Lagrange (Calcul. de Fonctions, Leçon X); and the latter was given by Euler in the Berlin Transactions.

(360.) From these formulæ we can immediately deduce exponential values for $\tan.x$. Dividing the second by the first, we find

$$\sqrt{-1}\tan.x = \frac{e^{x\sqrt{-1}} - e^{-x\sqrt{-1}}}{e^{x\sqrt{-1}} + e^{-x\sqrt{-1}}} = \frac{e^{2x\sqrt{-1}} - 1}{e^{2x\sqrt{-1}} + 1} \cdots\cdots [3].$$

(361.) From the formulæ [1] another celebrated formula may be immediately derived, called from its discoverer, ***Moivre's formula.*** Since

$$\cos.x + \sqrt{-1}\sin.x = e^{x\sqrt{-1}},$$

$$(\cos.x + \sqrt{-1}\sin.x)^m = e^{mx\sqrt{-1}}.$$

But also,

$$\cos.mx + \sqrt{-1}\sin.mx = e^{mx\sqrt{-1}};$$

$$\therefore (\cos.x + \sqrt{-1}\sin.x)^m = \cos.mx + \sqrt{-1}\sin.mx\,^* \cdots [4],$$

---

* In the case in which m is a rational number, this formula may be established on principles somewhat more elementary.

Let

$$\cos.x + \sqrt{-1}\sin.x = \text{x},$$

$$\therefore \cos.^2x - \sin.^2x + 2\sqrt{-1}\sin.x\cos.x = \text{x}^2,$$

$$\text{or } \cos.2x + \sqrt{-1}\sin.2x = \text{x}^2.$$

Multiplying this by the first, we obtain

$$\cos.x\cos.2x - \sin.x\sin.2x + \sqrt{-1}(\sin.x\cos.2x + \sin.2x\cos.x) = \text{x}^3,$$

$$\therefore \cos.3x + \sqrt{-1}\sin.3x = \text{x}^3,$$

first member cannot have more than n different values? To explain this, we should observe, that in the successive substitutions for mx in the second member of [4] the quantities added to mx are the successive multiples of $2m\pi$, so that after a certain number of substitutions, some multiple of $2m\pi$ will be introduced, which is also a multiple of 2π. As soon as this occurs, the corresponding sine and cosine will be the same as the sine and cosine of mx, and the value will be the same as that of the second member of [4]; and after this, every successive substitution will give the second member a value which it had obtained by a former substitution, and the values will be subject to no further variety. The first multiple of $2m\pi$, which will be also a multiple of 2π, will evidently be $2nm\pi$. For since m is a fraction whose denominator is n, the least integer, which, being multiplied into it will give an integral product, will be n. The nth substitution will therefore give the second member the same value which it has in [4], and there all variety of value ceases. There are then $n - 1$ substitutions for x, which give the second member $n - 1$ different values, and these, with the value in [4], complete the n values which the second member ought to have, in order perfectly to represent the first.

(362.) By multiplying the equations [2], the first member of the resulting equation is unity; from whence it follows that the formula

$$\cos.mx + \sqrt{-1}\sin.mx$$

is changed into its reciprocal by changing the sign of m, x, or the radical $\sqrt{-1}$.

(363.) The formulæ [2] suggest an extension of the formulæ for the sine and cosine of the sum of two angles to formulæ for the sine and cosine of the sum of any number of angles.

Let the angles be x', x'', x''', ... $x^{(n)}$; and let

$$\mathrm{x}' = \cos.x' + \sqrt{-1}\sin.x' = e^{x'\sqrt{-1}},$$
$$\mathrm{x}'' = \cos.x'' + \sqrt{-1}\sin.x'' = e^{x''\sqrt{-1}},$$
$$\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots$$
$$\mathrm{x}^{(n)} = \cos.x^{(n)} + \sqrt{-1}\sin.x^{(n)} = e^{x^{(n)}\sqrt{-1}}.$$

By taking the continued product of these, we obtain

$$\mathrm{x}'\mathrm{x}'' \ldots \mathrm{x}^{(n)} = e^{(x'+x''+\ldots x^{(n)})\sqrt{-1}}.$$

By [2] we have

$$\cos.(x'+x''\ldots x^{(n)}) + \sqrt{-1}\sin.(x'+x''+\ldots x^{(n)}) = e^{(x'+x''\ldots x^{(n)})\sqrt{-1}}.$$

Let $x' + x'' + x''' \ldots x^{(n)} = \Sigma x$. Hence we have

$$\mathrm{x}'\mathrm{x}''\mathrm{x}''' \ldots \mathrm{x}^{(n)} = \cos.\Sigma x + \sqrt{-1}\sin.\Sigma x \ldots [5].$$

If the continued product $\mathrm{x}'\mathrm{x}'' \ldots$ be developed, its terms will be some real and some imaginary; and by the principles of equations, the sum of the real terms must be equal to $\cos.\Sigma(x)$, and that of the imaginary to $\sqrt{-1}\sin.\Sigma(x)$. By examining the form of the development, we shall therefore obtain general expressions for the sine and cosine of the algebraical sum of any number of arcs, of which those for the sum and difference of two arcs in (43.) are particular cases.

The real terms of the development of $\mathrm{x}'\mathrm{x}'' \ldots \mathrm{x}^{(n)}$ are those in which the number of sines which enter as factors is found in the series

$$0,\ 2,\ 4,\ 6,\ \ldots$$

There is but one term corresponding to the first, which is the product of all the cosines. Let this be called c_n.

To the second there correspond the products of every combination of two sines, and these are to be multiplied respectively into the continued products of the remaining cosines. Since each sine is multiplied by $\sqrt{-1}$, the product

of every two will be necessarily negative. Let the sum of the continued products of every combination of two sines into the remaining cosines be expressed by $-c_{n-2}s_2$.

Since $(\sqrt{-1})^4 = +1$, the product of every four terms of the form $\sqrt{-1}\sin.x$ is positive. Let the sum of the continued products of every four sines into the remaining cosines be $c_{n-4}s_4$, and so on.

The imaginary terms of the development are those which involve an odd number of factors of the form $\sqrt{-1}\sin.x$. Let the sum of the continued products of every factor of this form into the remaining cosines be $\sqrt{-1}\,c_{n-1}s_1$.

Since $(\sqrt{-1})^3 = -\sqrt{-1}$, the sum of the products of every three factors of the form $\sqrt{-1}\sin.x$ into all the remaining cosines is $-\sqrt{-1}\,c_{n-3}s_3$, and these terms go on in the same manner alternately $+$ and $-$.

Hence we infer

$$\cos.\Sigma(x) = c_n - c_{n-2}s_2 + c_{n-4}s_4 - c_{n-6}s_6 \cdots\cdots \quad [6]$$

$$\sin.\Sigma(x) = c_{n-1}s_1 - c_{n-3}s_3 + c_{n-5}s_5 - \cdots\cdots\cdots \quad [7]$$

which are general formulæ for the sine and cosine of the algebraical sum of any number of arcs.

(364). From these formulæ we may easily deduce an expression for the tangent of the algebraical sum of any number of arcs of which the common formula for the sum or difference of two is a particular case.

Let the values of $\sin.\Sigma(x)$, $\cos.\Sigma(x)$ be substituted in

$$\tan.\Sigma(x) = \frac{\sin.\Sigma(x)}{\cos.\Sigma(x)}$$

and we find

$$\tan.\Sigma(x) = \frac{c_{n-1}s_1 - c_{n-3}s_3 + c_{n-5}s_5 \cdots\cdots}{c_n - c_{n-2}s_2 + c_{n-4}s_4 - \cdots\cdots}$$

Divide both numerator and denominator by c_n or the continued product of the cosines. Since $c_{n-1}s_1$ signifies the sum of the products of each sine into the continued product

of the remaining cosines, if it be divided by the continued product of all the cosines, the result will be the sum of the quotes of the sines divided by the cosines, that is the sum of the tangents. Let this be T_1.

In like manner, if $C_{n-2}S_2$ be divided by C_n the quote must be the sum of the products of every two tangents. Let this be T_2.

And in general if T_m signify the sum of the products of every m tangents we shall have

$$\tan.\Sigma(x) = \frac{T_1 - T_3 + T_5 - T_7 + \cdots\cdots}{1 - T_2 + T_4 - T_6 + \cdots\cdots} \qquad [8]$$

(365.) From the last three formulæ [6], [7], and [8], the first two of which do not appear to have been heretofore noticed, some elegant conclusions follow, when certain relations are instituted among the several angles. We shall confine ourselves here to a few examples of a very numerous class of inferences.

If there be but three arcs engaged in the formulæ, and that they be connected by the relation

$$x' + x'' + x''' = n\pi,$$

the formula [6] will become

$$\cos.n\pi = \cos.x' \cos.x'' \cos.x''' - \cos.x' \sin.x'' \sin.x''' - \cos.x'' \\ \sin.x \sin.x''' - \cos.x''' \sin.x' \sin.x''$$

adding to this the identity

$$0 = 3\cos.x' \cos.x'' \cos.x''' - 3\cos.x' \cos x'' \cos.x'''$$

we obtain

$$\cos.n\pi = 4\cos.x' \cos.x'' \cos.x''' - \cos.x' \cos.(x'' - x''') - \cos.x'' \\ \cos.(x''' - x') - \cos.x''' \cos.(x' - x'') \\ = 4\cos.x' \cos.x'' \cos.x''' - \cos.(x' + x'' - x''') \\ - \cos.(x' + x''' - x'') - \cos.(x'' + x''' - x');$$

from which by the relation between the three arcs we deduce

$$\cos.2x' + \cos.2x'' + \cos.2x''' = 4\cos.x' \cos.x'' \cos.x''' \pm 1,$$

since $\cos.n\pi = \pm 1$ according as n is even or odd.

By a process precisely similar we deduce from the formula [7] the relation

$$\sin.2x' + \sin.2x'' + \sin.2x''' = \mp \sin.x' \sin.x'' \sin.x'''$$

$-$ being taken when n is even and $+$ when n is odd.

Since

$$\tan.n\pi = 0$$

we immediately deduce from [8] the formulæ

$$\tan.x' + \tan.x'' + \tan.x''' = \tan.x' \tan.x'' \tan.x'''.$$

If $n=1$, $x'+x''+x'''=180°$. Hence it follows that "the product of the sines of the angles of a triangle, with its sign changed, is equal to the sum of the sines of their doubles, and that the product of the tangents of the angles is equal to the sum of the tangent of the angles themselves."

(366.) By Moivre's formula we can obtain immediately expressions for the sine, cosine and tangent, of a multiple arc in terms of the sine, cosine and tangent, of the simple arc. By supposing x alternately $+$ and $-$ we have

$$\cos.mx + \sqrt{-1}\sin.mx = (\cos.x + \sqrt{-1}\sin.x)^m$$

$$\cos.mx - \sqrt{-1}\sin.mx = (\cos.x - \sqrt{-1}\sin.x)^m$$

Hence

$$\cos.mx = \tfrac{1}{2}\left\{(\cos.x + \sqrt{-1}\sin.x)^m + (\cos.x - \sqrt{-1}\sin.x)^m\right\}$$

$$\sin.mx = \frac{1}{2\sqrt{-1}}\left\{(\cos.x + \sqrt{-1}\sin.x)^m - (\cos.x - \sqrt{-1}\sin.x)^m\right\}$$

$$\therefore \tan.mx = \frac{(\cos.x + \sqrt{-1}\sin.x)^m - (\cos.x - \sqrt{-1}\sin.x)^m}{(\cos.x + \sqrt{-1}\sin.x)^m + (\cos.x - \sqrt{-1}\sin.x)^m}$$

$$\therefore \sqrt{-1}\tan.mx = \frac{(1 + \sqrt{-1}\tan.x)^m - (1 - \sqrt{-1}\tan.x)^m}{(1 + \sqrt{-1}\tan.x)^m + (1 - \sqrt{-1}\tan.x)^m}$$

(367.) By the principles which have been just established, we are enabled to resolve the formula

$$z^{2m} - 2az^m + 1$$

into its simple factors when a is not greater than unity.

Let $a = \cos.x$, and the formula becomes

$$z^{2m} - 2\cos.x \,.\, z^m + 1.$$

By solving the equation

$$z'^{2m} - 2\cos.x z'^m + 1 = 0,$$

$$z'^m = \cos.\, x \pm \sqrt{-1}\,\sin.x,$$

$$\therefore z' = (\cos.x \pm \sqrt{-1}\,\sin.x)^{\frac{1}{m}},$$

$$\therefore z' = \cos.\frac{x}{m} \pm \sqrt{-1}\,\sin.\frac{x}{m}.$$

By the observations in (361.) it appears that of these two formulæ for z', each is susceptible of m different values found by substituting successively for x,

$$x,\ 2\pi + x,\ 4\pi + x,\ \cdots\ 2(m-1)\pi + x,$$

and therefore the proposed formula resolved into its simple factors will be

$$z^{2m} - 2z^m \cos.x + 1 =$$

$$\left(z - (\cos.\frac{x}{m} + \sqrt{-1}\sin.\frac{x}{m})\right)$$

$$\left(z - (\cos.\frac{x}{m} - \sqrt{-1}\sin.\frac{x}{m})\right)$$

$$\left(z - (\cos.\frac{2\pi + x}{m} + \sqrt{-1}\sin.\frac{2\pi + x}{m})\right)$$

$$\left(z - (\cos.\frac{2\pi + x}{m} - \sqrt{-1}\sin.\frac{2\pi + x}{m})\right)$$

$$\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot$$

$$\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot$$

$$\left(z - (\cos.\frac{2(m-1)\pi + x}{m} + \sqrt{-1}\sin.\frac{2(m-1)\pi + x}{m})\right)$$

$$\left(z - (\cos.\frac{2(m-1)\pi + x}{m} - \sqrt{-1}\sin.\frac{2(m-1)\pi + x}{m})\right)$$

If each pair of simple factors be united by multiplication, we shall have the proposed formula resolved into real quadratic factors,

$$z^{2m} - 2z^m \cos.x + 1 =$$

$$(z^2 - 2z\cos.\frac{x}{m} + 1)(z^2 - 2z\cos.\frac{2\pi+x}{m} + 1)$$

$$\times(z^2 - 2z\cos.\frac{4\pi+x}{m} + 1)(z^2 - 2z\cos.\frac{6\pi+x}{m} + 1)$$

. . . .

. . . .

$$\times(z^2 - 2z\cos.\frac{2(m-1)\pi+x}{m} + 1).$$

It appears, therefore, that in general all the simple factors are imaginary, and the quadratic real. This theorem was discovered by *Moivre*, and previously another similar to it, and deducible from it, was discovered by *Cotes*. In the last, let $x = 0$, and we have

$$z^{2m} - 2z^m + 1 = (z^2 - 2z + 1)\times(z^2 - 2z\cos.\frac{2\pi}{m} + 1)$$

$$\times(z^2 - 2z\cos.\frac{4\pi}{m} + 1)$$

. . . .

. . . .

$$\times(z^2 - 2z\cos.\frac{2(m-1)\pi}{m} + 1).$$

This may be somewhat simplified; but to effect this, it is necessary to consider distinctly the cases where m is odd and even.

1°. *Let* m *be odd* $\because m - 1$ is even. Hence, exclusive of $z^2 - 2z + 1$, there are an even number $(m - 1)$ of quadratic factors.

But

$$\cos.\frac{2(m-1)\pi}{m} = \cos.\left(2\pi - \frac{2\pi}{m}\right) = \cos.\frac{2\pi}{m}.$$

Hence the first and last factors in the above column are identical, and when united, give

$$(z^2 - 2z\cos.\frac{2\pi}{m} + 1)^2;$$

in like manner the second and penultimate, and every pair of factors equidistant from the first and last are equal, and when united, give, in the same manner, squares. Since the number of factors in this column $(m-1)$ is even, there will be two in the middle which, united, give

$$(z^2 - 2z\cos.\frac{(m-1)\pi}{m} + 1)^2.$$

Also, since

$$z^{2m} - 2z^m + 1 = (z^m - 1)^2,$$
$$z^2 - 2z + 1 = (z - 1)^2;$$

we find by taking the square root of both members,

$$\begin{aligned} z^m - 1 = (z-1) &\times (z^2 - 2z\cos.\frac{2\pi}{m} + 1) \\ &\times (z^2 - 2z\cos.\frac{4\pi}{m} + 1) \\ &\quad \cdots\cdots\cdots \\ &\quad \cdots\cdots\cdots \\ &\times (z^2 - 2z\cos.\frac{(m-1)\pi}{m} + 1). \end{aligned}$$

2°. *If* m *be even,* and $\because m-1$ odd, there will be a solitary factor in the middle of the column, which will be

$$\begin{aligned} z^2 - 2z\cos.\pi + 1 &= z^2 + 2z + 1 \\ &= (z+1)^2. \end{aligned}$$

The first and last factor, and those equidistant from them being united as before, and the roots of both members taken, we have

$$z^m - 1 = (z - 1) \times (z^2 - 2z\cos.\frac{2\pi}{m} + 1)$$
$$\times (z^2 - 2z\cos.\frac{4\pi}{m} + 1)$$
$$\cdots\cdots\cdots$$
$$\times (z^2 - 2z\cos.\frac{2\left(\frac{m}{2} - 2\right)\pi}{m} + 1)$$
$$\times (z + 1),$$

which are the factors of $z^m - 1$ when m is even *.

(368.) It is plain, therefore, that when m is odd, $z^m - 1$ has but one simple factor, which is real, scil. $z - 1$, all the others being imaginary; and when m is even, it has two real simple factors, scil. $z - 1$ and $z + 1$, all the others being imaginary.

The investigation of the simple factors of $z^m - 1$ determines the values of the mth roots of unity; for if $z - a$ be a factor of $z^m - 1$, a must be a root of the equation

$$z^m - 1 = 0,$$

the roots of which are obviously the mth roots of unity. Hence it appears that the roots of unity are included in the general formula

$$\cos.\frac{2n\pi}{m} + \sqrt{-1}\sin.\frac{2n\pi}{m},$$

and are found by substituting successively for n the terms of the series

$$0,\ 1,\ 2,\ 3,\ \ldots.$$

Thus the cube roots of unity are

$$z = 1,$$
$$z = \cos.120^\circ + \sqrt{-1}\sin.120^\circ = \frac{-1+\sqrt{-3}}{2},$$
$$z = \cos.240^\circ + \sqrt{-1}\sin.240^\circ = \frac{-1-\sqrt{-3}}{2}.$$

---

* Another investigation of these theorems is given in my Geometry (588.)

The last two roots may easily be verified by raising them to the third power.

In like manner, by substituting 4 for m, and 0, 1, 2, 3, for n, we find the fourth roots of unity to be

$$z = + \sqrt{-1}, \quad z = -1, \quad z = -\sqrt{-1}, \quad z = +1.$$

Having determined the factors of $x^m - 1$, we may find those of $z^m - a^m$ for

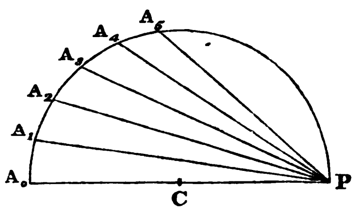
$$z^m - a^m = a^m\left(\frac{z^m}{a^m} - 1\right).$$

The m factors of $\frac{z^m}{a^m} - 1$ being found, each of them multiplied by a is the corresponding factor of $z^m - a^m$; and for the same reason, the determination of the mth roots of unity involves that of the mth roots of a^m.

(369.) The principles which have been established in this section supply a very elegant geometrical construction representing the sums of the squares, cubes, &c. of the roots of a quadratic equation of the form

$$z^2 - pz + 1 = 0,$$

in which p is not > 2.



Let a circle be described with a radius equal to unity, and let $A_0PA_1 = x$, and let the arcs A_0A_1, A_1A_2, A_2A_3, ... be equal.

It is obvious that

$$PA_1 = 2\cos.x, \quad PA_2 = 2\cos.2x, \quad PA_3 = 2\cos.3x, \text{ \&c.}$$

Let ω be assumed, so that $PA_1 = p$, which is always possible, since p is not > 2. Hence

$$z^2 - 2z\cos.x + 1 = 0,$$

$$\therefore z + \frac{1}{z} = 2\cos.x = PA_1,$$

and by (359.), it follows that

$$z^2 + \frac{1}{z^2} = 2\cos.2x = PA_2$$

$$z^3 + \frac{1}{z^3} = 2\cos.3x = \text{PA}_3,$$

$$z^4 + \frac{1}{z^4} = 2\cos.4x = \text{PA}_4,$$

. . . .

. . . .

The roots of the equation proposed being reciprocals, it is obvious that the first members of these equations are the sums of their successive powers.

SECTION II.

Of the development of the sines and cosines of multiple arcs in powers of the sines and cosines of the simple arcs.

(370.) Notwithstanding the elementary nature of the trigonometrical analysis, and the attention which has been devoted to its various details from the time of Euler to the present day by the greatest mathematicians, yet the analysis of angular sections remained until a late period in a most imperfect state. Formulæ expressing relations between the sine and cosine of an arc and those of its multiples were established by Euler, and subsequently confirmed by the searching analysis of Lagrange, which have since been proved inaccurate, or true only under particular conditions, and it was only within the last three years that the complete exposition of this theory has been published, and general formulæ assigned expressing those relations. In the year 1811, *Poisson* detected an error in a formula of Euler, expressing the relation between the power of the sine or cosine of an arc, and the sines and cosines of certain multiples of the same arc. But the most complete dis-

cussion of this subject which has hitherto appeared is contained in a memoir read before the Academy of Sciences at *Paris* by *Poinsot* *, an eminent French mathematician, in the year 1823, and further developed by him in another memoir published in the present year (1825).

The developments respecting multiple arcs may be divided into two distinct classes. The first includes all series in which the sine or cosine of a multiple arc is expressed in powers of those of the simple arc, and the second those in which a power of the sine or cosine of a simple arc is expressed in a series of sines or cosines of its multiples; to the former we shall devote the present section, reserving the latter for the following one.

The series in powers of the sine, cosine, &c. may be either ascending or descending, and accordingly the several problems into which our analysis resolves itself may be enumerated as follow:

To develop, 1°. $\left.\begin{array}{r}\cos.mx\\ \sin.mx\end{array}\right\}$ in ascending powers of $\cos.x$.

2°. $\left.\begin{array}{r}\sin.mx\\ \cos.mx\end{array}\right\}$ in ascending powers of $\sin.x$.

3°. $\left.\begin{array}{r}\sin.mx\\ \cos.mx\end{array}\right\}$ in ascending powers of $\tan.x$.

4°. $\left.\begin{array}{r}\cos.mx\\ \sin.mx\end{array}\right\}$ in descending powers of $\cos.x$.

5°. $\left.\begin{array}{r}\sin.mx\\ \cos.mx\end{array}\right\}$ in descending powers of $\sin.x$.

---

* This mathematician has rendered himself distinguished by the invention of the " theory of couples" (*Theorie des couples*), a most powerful instrument of investigation in analytical mechanics, and one which has not yet received the attention which it deserves from mathematical writers either here or on the continent, and which I venture to predict it must ultimately command.

PROP. XCIX.

(371.) *To develop* cos.mx *in a series of ascending powers of* cos.x.

Let $\cos x = y$, and let

$$z = y + \sqrt{y^2 - 1},$$

$$\therefore \frac{1}{z} = y - \sqrt{y^2 - 1}.$$

But by (359.),

$$2\cos.mx = z^m + \frac{1}{z^m}.$$

If then z^m be obtained in ascending powers of y, and z^{-m} deduced from it by changing the sign of m, we shall thence obtain $2\cos.mx$ in a series of the required form.

Let

$$z^m = u = A_0 + A_1 y + A_2 y^2 + A_3 y^3 + \cdots$$

The solution of the question will be effected if the values of the coefficients of this series can be obtained without introducing any condition which restricts the generality of the problem.

Let the series assumed to express u be twice differentiated, and the results will be

$$\frac{du}{dy} = A_1 + 2A_2 y + 3A_3 y^2 + 4A_4 y^3 + \cdots$$

$$\frac{d^2u}{dy^2} = 2A_2 + 2.3A_3 y + 3.4A_4 y^2 + \cdots$$

Also, let

$$u = (y + \sqrt{y^2 - 1})^m$$

be twice successively differentiated, and the results are

$$\left(\frac{du}{dy}\right)^2 (y^2 - 1) - m^2 u = 0,$$

$$\left(\frac{du}{dy}\right)\left(\frac{d^2u}{dy^2}\right)(y^2 - 1) + \left(\frac{du}{dy}\right)^2 y - \left(\frac{du}{dy}\right) m^2 u = 0,$$

which divided by $\frac{du}{dy}$, gives

$$\frac{d^2u}{dy^2}(y^2-1)+\left(\frac{du}{dy}\right)y-m^2u=0.$$

Let the values of u, $\frac{du}{dy}$, $\frac{d^2u}{dy^2}$, derived from differentiating the assumed series, be substituted in the last equation, and let the result be arranged according to the ascending powers of y. We shall thus obtain the following series:

$$\begin{aligned}
&\mathrm{A}_0m^2+2\mathrm{A}_2,\\
&+[\mathrm{A}_1(m^2-1)+2.3\mathrm{A}_3]y,\\
&+[\mathrm{A}_2(m^2-4)+3.4\mathrm{A}_4]y^2,\\
&+[\mathrm{A}_3(m^2-9)+4.5\mathrm{A}_5]y^3,\\
&+[\mathrm{A}_4(m^2-16)+5.6\mathrm{A}_6]y^4.\\
&\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\\
&\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\\
&+\{\mathrm{A}_{n-2}[m^2-(n-2)^2]+(n-1)(n)\mathrm{A}_n\}y^{n-2},\\
&\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\\
&\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ =0.
\end{aligned}$$

Since this must be fulfilled independently of y, the coefficients must severally $=0$. Hence we find

$$\begin{aligned}
\mathrm{A}_2&=-\frac{m^2}{2}\mathrm{A}_0,\\
\mathrm{A}_3&=-\frac{m^2-1}{2.3}\mathrm{A}_1,\\
\mathrm{A}_4&=-\frac{m^2-4}{3.4}\mathrm{A}_2,\\
\mathrm{A}_5&=-\frac{m^2-9}{4.5}\mathrm{A}_3,\\
&\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\\
&\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\\
\mathrm{A}_n&=-\frac{m^2-(n-2)^2}{(n-1)n}\mathrm{A}_{n-2},\\
\cdot\ &=\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot
\end{aligned}$$

Hence we obtain the following conditions:

$$A_2 = -\frac{m^2}{2}A_0,$$

$$A_3 = -\frac{m^2-1}{2.3}A_1,$$

$$A_4 = +\frac{m^2(m^2-4)}{2.3.4}A_0,$$

$$A_5 = +\frac{(m^2-1)(m^2-9)}{2.3.4.5}A_1,$$

$$A_6 = -\frac{m^2(m^2-4)(m^2-16)}{2.3.4.5.6}A_0,$$

. . . .

. . . .

The law of which is evident. These conditions, however, fail to determine the first two coefficients A_0, A_1. To find these, let $y = 0$ in the series for u and $\frac{du}{dy}$, and also in the values

$$u = z^m = (y + \sqrt{y^2 - 1})^m,$$

$$\frac{du}{dy} = \frac{mu}{\sqrt{y^2-1}};$$

and equating the results, we obtain

$$A_0 = (\sqrt{-1})^m = (-1)^{\frac{m}{2}},$$

$$A_1 = m(\sqrt{-1})^{m-1} = m(-1)^{\frac{m-1}{2}},$$

whence we find

$$A_2 = -\frac{m^2}{2}(-1)^{\frac{m}{2}},$$

$$A_3 = -\frac{m^2-1^2}{2.3}\cdot m(-1)^{\frac{m-1}{2}},$$

$$A_4 = +\frac{m^2(m^2-2^2)}{2.3.4}(-1)^{\frac{m}{2}},$$

$$A_5 = +\frac{(m^2-1^2)(m^2-3^2)}{2.3.4.5}\cdot m(-1)^{\frac{m-1}{2}},$$

$$A_6 = -\frac{m^2(m^2-2^2)(m^2-4^2)}{2.3.4.5.6}(-1)^{\frac{m}{2}},$$

$$. = - \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .$$

$$. = + \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .$$

Hence we find

$$z^m=(-1)^{\frac{m}{2}}\left\{1-\frac{m^2}{1.2}y^2+\frac{m^2(m^2-2^2)}{1.2.3.4}y^4-\frac{m^2(m^2-2^2)(m^2-4^2)}{1.2.3.4.5.6}y^6\right.$$
$$\left.+\frac{m^2(m^2-2^2)(m^2-4^2)(m^2-6^2)}{1.2.3.4.5.6.7.8}y^8-\ldots\right\}$$
$$+m(-1)^{\frac{m-1}{2}}\left\{y-\frac{(m^2-1^2)}{1.2.3}y^3+\frac{(m^2-1^2)(m^2-3^2)}{1.2.3.4.5}y^5\right.$$
$$\left.-\frac{(m^2-1^2)(m^2-3^2)(m^2-5^2)}{1.2.3.4.5.6.7}y^7+\ldots\right\}$$

To find the series for z^{-m}, it is only necessary to change the sign of m in the result which has just been obtained. Since neither of the series in this result contains any odd power of m, this change produces no other effect than to change the sign of the coefficient of the second parenthesis. Let the series in the first parenthesis be called for brevity s, and that in the second s$'$, and we have

$$z^m = (-1)^{\frac{m}{2}}.\,\mathrm{s} + m(-1)^{\frac{m-1}{2}}.\,\mathrm{s}',$$

$$z^{-m} = (-1)^{-\frac{m}{2}}.\,\mathrm{s} + m(-1)^{-\frac{m-1}{2}}.\,\mathrm{s}';$$

since $-m(-1)^{-\frac{m-1}{2}} = m(-1)^{-\frac{m-1}{2}}$.

Hence, by addition we obtain,

$$z^m+z^{-m}=[(-1)^{\frac{m}{2}}+(-1)^{-\frac{m}{2}}]\mathrm{s}+[(-1)^{\frac{m-1}{2}}+(-1)^{-\frac{m-1}{2}}]m\mathrm{s}',$$

$$\because 2\cos.mx=[(-1)^{\frac{m}{2}}+(-1)^{-\frac{m}{2}}]\mathrm{s}+[(-1)^{\frac{m-1}{2}}$$
$$+(-1)^{-\frac{m-1}{2}}]m\mathrm{s}' \ldots . [1],$$

which is the development sought.

(372.) The form of the coefficients of this formula may be changed. By (361.), we have

$(\cos.x + \sqrt{-1}\sin.x)^m = \cos.m(2n\pi \pm x) + \sqrt{-1}\sin.m(2n\pi \pm x)$

n being any positive integer. Let $x = \frac{1}{2}\pi$, $\because$

$$(-1)^{\frac{m}{2}}=\cos.\tfrac{1}{2}m(4n\pm 1)\pi+\sqrt{-1}\sin.\tfrac{1}{2}m(4n\pm 1)\pi,$$

$$\therefore (\sqrt{-1})^{-m}=\cos.\tfrac{1}{2}m(4n\pm 1)\pi-\sqrt{-1}\sin.\tfrac{1}{2}m(4n\pm 1)\pi,$$

$$\therefore (-1)^{\frac{m}{2}}+(-1)^{-\frac{m}{2}}=2\cos.\tfrac{1}{2}m(4n\pm 1)\pi,$$

$$(-1)^{\frac{m-1}{2}}+(-1)^{-\frac{m-1}{2}}=2\cos.\tfrac{1}{2}(m-1)(4n\pm 1)\pi.$$

Hence the series for $\cos.mx$ becomes

$$\cos.mx=\cos.\tfrac{1}{2}m(4n\pm 1)\pi.s+\cos.\tfrac{1}{2}(m-1)(4n\pm 1)\pi.ms'...[2].$$

In this formula n is an indeterminate integer for each value of which the second member has two values corresponding to the double sign $\pm$. The successive terms of the series

$$0,\ 1,\ 2,\ 3,\ \cdot\ \cdot\ \cdot\ \cdot$$

being substituted for n in $\cos.\frac{1}{2}m(4n+1)\pi$, it will successively assume different values until the number substituted for n is equal to the denominator of m; for this value of n the value of $\cos.\frac{1}{2}m(4n+1)\pi$ will be equal to that obtained by substituting 0 for n; and all integers greater than the denominator of m will in like manner give a constant repetition of values before obtained by substituting for n values less than the denominator of m. It follows, therefore, that $\cos.\frac{1}{2}m(4n+1)\pi$ is in general susceptible of as many different values as there are units in the denominator of m, and no more. In like manner $\cos.\frac{1}{2}m(4n-1)\pi$ is susceptible of the same number of values, and therefore the coefficient of s is susceptible of twice as many values as there are units in the denominator of m, and a like observation applies to the coefficient of ms'.

Since s and s$'$ involve no functions of x, except $\cos.x$, the change of x into $2n\pi\pm x$ makes no change in their value, and it follows, therefore, that for a given value of $\cos.x$ the second member of [2] is susceptible of twice as many values as there are units in the denominator of m. It is therefore necessary to show how $\cos.mx$ can have several values cor-

responding to a given value of $\cos.x$. The angle x being changed into $2n'\pi \pm x$, n' being an integer, makes no change in $\cos.x$, but changes $\cos.mx$ into $\cos.m(2n'\pi \pm x)$, which by (361.) has twice as many values as there are units in the denominator of m. Hence the formula [2] will be more generally and correctly expressed thus,

$$\cos.m(2n'\pi \pm x) = \cos.\tfrac{1}{2}m(4n \pm 1)\pi \,.\, s$$
$$+\cos.\tfrac{1}{2}(m-1)(4n \pm 1)\pi \,.\, ms',$$

where both members have the same number of values, and where the values of the indeterminate integers n', n are supposed to be less than the denominator of m.

It still remains, however, to show the values of each member which correspond respectively to those of the other. Since the value of each member changes by ascribing different values to the integers n' and n, this question only amounts to the determination of the relation between any two corresponding values of these integers.

Let $x = \frac{1}{2}\pi$, and therefore $s = 1$, $s' = 0$. Hence

$$\cos.m(2n'\pi \pm \tfrac{1}{2}\pi) = \cos.\tfrac{1}{2}m(4n \pm 1)\pi,$$
$$\text{or } \cos.\tfrac{1}{2}m(4n' \pm 1)\pi = \cos.\tfrac{1}{2}m(4n \pm 1)\pi.$$

Since n and n' are not supposed to receive any value greater than the denominator of m (for all the values of the cosine after that would only be repetitions of former values), this last condition can only be satisfied by

$$n = n'.$$

Hence the formula becomes *

* In clearing the formula [1] of imaginary quantities, *Lagrange* has fallen into an error which was lately detected by *Poinsot,* and the difficulty explained as above. *Lagrange's* mistake arose from assuming that

$$(\sqrt{-1})^m = \cos.\tfrac{1}{2}m\pi + \sqrt{-1}\sin.\tfrac{1}{2}m\pi,$$

which is evidently erroneous, since the first member has as many different values as there are units in the denominator of m, and

$$\cos.m(2n\pi \pm x) = \cos.\tfrac{1}{2}m(4n \pm 1)\pi \,.\, \mathrm{s}$$
$$+ \cos.\tfrac{1}{2}(m-1)(4n \pm 1)\pi \,.\, m\mathrm{s}' \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad [3].$$

(373.) It does not always happen that the formula expressing the value of $\cos.mx$ includes both terms of the second member; for the angles whose cosines are the multipliers of s and s′ in [3] may one or other of them be an odd multiple of a right angle, in which case the multiplier will be $= 0$, and the term will disappear.

To determine the conditions under which this can occur, it is necessary to consider when either of the numbers

$$\tfrac{1}{2}m(4n \pm 1)\pi, \quad \tfrac{1}{2}(m-1)(4n \pm 1)\pi,$$

is an exact *odd* multiple of $\frac{1}{2}\pi$. This evidently takes place when either of the numbers

$$m(4n \pm 1), \quad (m-1)(4n \pm 1),$$

is an odd integer.

Let $m = \dfrac{m'}{n'}$, and let I be any odd integer. That the first of the above numbers be an odd integer, it is necessary that

$$m'(4n \pm 1) = n'\mathrm{I}.$$

Since m' and n' are prime, one or other must be an odd number; but since $4n \pm 1$ and I are also odd, it is necessary that both m' and n' should be odd.

Also, since m' is prime to n', and measures $n'\mathrm{I}$, it must

---

the second member has but one value. He forgot to take into account, that while the change of x into $2n\pi + x$ produces no change on

$$(\cos.x + \sqrt{-1}\sin.x)^m,$$

it does produce a change on

$$\cos.mx + \sqrt{-1}\sin.mx.$$

In fact, without this consideration, *Moivre's formula* itself is involved in the absurdity of one member having a greater number of different values than the other.

measure I. Let $\frac{\text{I}}{m'} = i$, which must be an odd integer, since both I and m' are odd. Hence

$$4n \pm 1 = n'i,$$

$$\therefore \frac{4n \pm 1}{n'} = i.$$

But since n is supposed to receive no value greater than n', i cannot be greater than 4; and since it is an odd integer, it must be either 1 or 3. The two corresponding values of n are

$$n = \frac{n' \mp 1}{4},\ n = \frac{3n' \mp 1}{4}.$$

The denominator n' being odd, must be either of the form $4t + 1$, or $4t - 1$.

If n' be of the form $4t + 1$, the two values of n must be

$$n = \frac{n' - 1}{4},\ n = \frac{3n' + 1}{4};$$

since 4 evidently would not measure $n' + 1 = 4t + 2$, nor $3n' - 1 = 12t + 3 - 1 = 12t + 2$.

These values of n being substituted in [3], and m being changed into $\frac{m'}{n'}$, and the sign + only being used for the first, and − for the second, give

$$\left.\begin{aligned} \cos.\frac{m'}{n'}\left(\frac{n'-1}{2}\pi + x\right) &= \cos.\tfrac{1}{2}m'\pi \,.\, \mathrm{s} \\ &+ \cos.\tfrac{1}{2}(m' - n')\pi \,.\, \frac{m'}{n'}\mathrm{s}' \\ \cos.\frac{m'}{n'}\left(\frac{3n'+1}{2}\pi - x\right) &= \cos.\tfrac{3}{2}m'\pi \,.\, \mathrm{s} \\ &+ \cos.\tfrac{3}{2}(m' - n')\pi \,.\, \frac{m'}{n'}\mathrm{s}' \end{aligned}\right\} \cdots\cdots [4].$$

Since m' and n' are odd,

$$\cos.\tfrac{1}{2}m'\pi = 0,\quad \cos.\tfrac{1}{2}(m' - n')\pi = \pm 1,$$
$$\cos.\tfrac{3}{2}m'\pi = 0,\quad \cos.\tfrac{3}{2}(m' - n')\pi = \pm 1,$$

$$\left.\begin{array}{l}\because \cos.\dfrac{m'}{n'}\left(\dfrac{n'-1}{2}\pi + x\right) = \pm \dfrac{m'}{n'}s' \\ \cos.\dfrac{m'}{n'}\left(\dfrac{3n'+1}{2}\pi - x\right) = \pm \dfrac{m'}{n'}s'\end{array}\right\} \quad \cdots\cdots [5],$$

the sign + being used when $\frac{1}{2}(m' - n')$ is even, and − when odd.

If n' be of the form $4t - 1$, the two values of n are

$$n = \frac{n'+1}{4}, \; n = \frac{3n'-1}{4},$$

for it is evident that 4 would not in this case measure $n'-1$, or $3n' + 1$.

These values being substituted in [3], and m being changed as before into $\dfrac{m'}{n'}$, we obtain

$$\left.\begin{array}{l}\cos.\dfrac{m'}{n'}\left(\dfrac{n'+1}{2}\pi - x\right) = \cos.\frac{1}{2}m'\pi \,.\, s \\ \qquad + \cos.\frac{1}{2}(m' - n')\pi \,.\, \dfrac{m'}{n'}s' \\ \cos.\dfrac{m'}{n'}\left(\dfrac{3n'-1}{2}\pi + x\right) = \cos.\frac{3}{2}m'\pi \,.\, s \\ \qquad + \cos.\frac{3}{2}(m' - n')\pi \,.\, \dfrac{m'}{n'}s'\end{array}\right\} \quad \cdots\cdots [6],$$

Hence as before, we find

$$\left.\begin{array}{l}\cos.\dfrac{m'}{n'}\left(\dfrac{n'+1}{2}\pi - x\right) = \pm \dfrac{m'}{n'}s' \\ \cos.\dfrac{m'}{n'}\left(\dfrac{3n'-1}{2}\pi + x\right) = \pm \dfrac{m'}{n'}s'\end{array}\right\} \quad \cdots\cdots [7].$$

The signs + and − being used as before.

(374.) The condition under which

$$(m - 1)(4n \pm 1) = \frac{m'-n'}{n'}(4n \pm 1)$$

should be an odd integer, may be immediately derived from those of the last case by changing m' into $m' - n'$. Hence the two values of n are the same as those already found,

and n' and $m' - n'$, must be odd integers. Hence m' is even. Hence we have

$$\cos.\tfrac{1}{2}m'\pi = \pm 1, \qquad \cos.\tfrac{1}{2}(m' - n')\pi = 0,$$
$$\cos.\tfrac{3}{2}m'\pi = \pm 1, \qquad \cos.\tfrac{3}{2}(m' - n')\pi = 0.$$

Hence the formulæ [4] and [6] become

$$\left.\begin{aligned}
&\cos.\frac{m'}{n'}\left(\frac{n'-1}{2}\pi + x\right) = \pm\,\mathrm{s}\\
&\cos.\frac{m'}{n'}\left(\frac{3n'+1}{2}\pi - x\right) = \pm\,\mathrm{s}\\
&\cos.\frac{m'}{n'}\left(\frac{n'+1}{2} \quad - x\right) = \pm\,\mathrm{s}\\
&\cos.\frac{m'}{n'}\left(\frac{3n'-1}{2} \quad + x\right) = \pm\,\mathrm{s}
\end{aligned}\right\} \quad [8]$$

the sign + being used when $\frac{1}{2}m'$ is even and − when odd.

(375.) From the preceding observations it appears that when the denominator of m is odd there are always two values of an angle x whose cosine is given, of which the cosine of the multiple mx admits of being expressed by a single series of ascending powers of the given cosine *, but that for all other values of the arc whose cosine is given the cosine of the same multiple requires the combination of both series s and s'.

If the denominator of m be even, there is no value whatever of the angle whose cosine is given which allows of $\cos.mx$ being expressed by a single series.

(376.) The case in which m is an integer comes under the cases where the denominator of m is of the form $4t + 1$, t being in this case $= 0$. If m be odd we have by [5]

$$\cos.mx = \pm\, m\mathrm{s}',$$

---

* Before the publication of Poinsot's Memoir these cases were not noticed. *Lagrange* expressly states that whenever m is a fraction both terms of the second member of [3] are *necessary*.

the sign + being used when $\frac{1}{2}(m-1)$ is even and $-$ when odd. If m be even we have by the first of [8]

$$\cos. mx = \pm \text{s},$$

the sign + being used when $\frac{1}{2}m$ is even and $-$ when odd.

(377.) The laws of the two series s and s$'$ are easily defined. Let T be the rth term of s and T$'$ of s$'$; by attending to the forms of the coefficients and exponents we find

$$\text{T} = \pm \frac{m^2(m^2-2^2)(m^2-4^2)(m^2-6^2)\cdots(m^2-(2r-4)^2)}{1.2.3. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad 2(r-1)} y^{2(r-1)}$$

$$\text{T}' = \pm \frac{(m^2-1^2)(m^2-3^2)(m^2-5^2)\cdots(m^2-(2r-3)^2)}{1.2.3. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad 2r-1} y^{2r-1}.$$

It is evident from the forms of these terms that the series s can only terminate when m is an even integer, and s$'$ when m is an odd integer.

(378.) To determine the number of terms in each series when it is finite, let n be the sought number. The $(n+1)$th term must therefore $=0$. Substituting $n+1$ for r in T and T$'$, and putting the results $=0$, we obtain

$$m^2-(2n+2-4)^2=0,$$

$$\because n = \frac{m}{2}+1,$$

the number of terms in s; and

$$m^2-(2n+2-3)^2=0,$$

$$\because n = \frac{m+1}{2},$$

the number of terms in s$'$.

(379.) To obtain the last term (z) of s, it is only necessary to substitute the value of n in place of r in T, and the result is

$$z = \pm \frac{m^2(m^2-2^2)(m^2-4^2)\cdots\cdots(m^2-(m-2)^2)}{1.2.3. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad m} y^m.$$

Each factor of the numerator may be resolved into two thus,

$$m^2 = m \times m,$$

$$(m^2-2^2) = (m+2) \times (m-2),$$

$$(m^2 - 4^2) = (m + 4) \times (m - 4),$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$(m^2 - (m - 2)^2) = (2m - 2) \times 2.$$

The second factors of these, beginning from the lowest, are obviously the even integers from 2 to m inclusive, and the first factors beginning from the highest are the even integers from m to $2m-2$ inclusive. Thus the simple factors of the numerator are all the even integers from 2 to $2m - 2$ inclusive, the factor m being twice repeated. The numerator of x may therefore be written thus,

$$2.4.6. \quad \cdot \quad \cdot \quad \cdot \quad (2m - 2) \times m,$$

which is equivalent to

$$1.2.3. \quad \cdot \quad \cdot \quad \cdot \quad (m - 1) \times m \times 2^{m-1}.$$

The factors of the denominator destroying all these except 2^{m-1}, we have

$$x = \pm 2^{m-1} y^m,$$

$+$ being taken when $\frac{m}{2}$ is even and $-$ when odd.

(380.) To determine the last term x' of s', let $\frac{m+1}{2}$ be substituted for r in the general term and we obtain

$$x' = \pm \frac{(m^2-1^2)(m^2-3^2) \quad \cdot \quad \cdot \quad \cdot \quad (m^2-(m-2)^2)}{1.2.3 \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad m} y^m.$$

Each of the factors of the numerator may, as before, be resolved into two thus,

$$(m^2 - 1^2) = (m+1) \times (m-1),$$

$$(m^2 - 3^2) = (m+3) \times (m-3),$$

$$\cdot \quad \cdot \quad \cdot \quad = \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\cdot \quad \cdot \quad \cdot \quad = \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$(m^2 - (m-2)^2) = (2m-2) \times 2.$$

The last factors of each of these beginning from the lowest are the even integers from 2 to $m - 1$ inclusive, and the first beginning from the highest are the even integers from $m + 1$

to $2m-2$ inclusive. Hence the factors of the numerator may be expressed thus,

$$\begin{aligned} &2.4.6. \quad \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \quad (2m-2) \\ &= 1.2.3. \quad \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \quad (m-1).2^{m-1}. \end{aligned}$$

Hence

$$s' = \pm \frac{1}{m} 2^{m-1} y^m.$$

PROP. C.

(381.) *To develop sin.*mx *in ascending powers of cos.*x.

By subtracting the value of z^{-m} obtained in (371.) from that of z^m, and the result being disengaged from the imaginary symbols by the method used in (372.) becomes

$$\sin.m(2n\pi \pm x) = \sin.\tfrac{1}{2}m(4n\pm1)\pi s + \sin.\tfrac{1}{2}(m-1)(4n\pm1)\pi.ms'. \quad [9]$$

All the observations in the last proposition are equally applicable here. When the denominator of m is an odd integer there are always two values of an angle x whose cosine is given, which are such that $\sin.mx$ will be expressed by only one of the two series in [9].

(382.) To determine the conditions under which this will happen it is necessary to determine when either of the numbers

$$m(4n \pm 1) \qquad (m-1)(4n \pm 1)$$

is an even integer.

To find the values of n which will render $m(4n \pm 1)$ an even integer, let

$$m(4n \pm 1) = \text{I},$$

$$\therefore m'(4n \pm 1) = \text{I}n',$$

$$\therefore 4n \pm 1 = \frac{\text{I}}{m'}n'.$$

Hence $\frac{\text{I}}{m'}n'$ is an odd integer, therefore $\frac{\text{I}}{m'}$ must be an odd

integer, therefore m' must be even. Let $\frac{1}{m'} = i$,

$$\therefore 4n \pm 1 = in'.$$

It may be proved, as in the former case, that i must be either 1 or 3, and that when n' has the form $4t + 1$, the values of n are

$$n = \frac{n'-1}{4},\ n = \frac{3n'+1}{4};$$

and when n' has the form $4t - 1$, the values are

$$n = \frac{n'+1}{4},\ n = \frac{3n'-1}{4}.$$

(383.) In like manner, in order that $(m - 1)(4n \pm 1)$ be an even integer, the same values of n are obtained, and it is necessary that n' should be odd, and $m' - n'$ even, and therefore m' odd.

(384.) Hence if m' be even, and the values of n obtained above be substituted for it in [9], we obtain

$$\left.\begin{aligned}
\sin.\frac{m'}{n'}\left(\frac{n'-1}{2}\pi + x\right) &= \sin.\tfrac{1}{2}m'\pi\,.\,s \\
&\quad + \sin.\tfrac{1}{2}(m' - n')\pi\,.\,\frac{m'}{n'}s' \\
\sin.\frac{m'}{n'}\left(\frac{3n'+1}{2}\pi - x\right) &= \sin.\tfrac{3}{2}m'\pi\,.\,s \\
&\quad + \sin.\tfrac{3}{2}(m' - n')\pi\,.\,\frac{m'}{n'}s' \\
\sin.\frac{m'}{n'}\left(\frac{n'+1}{2}\pi - x\right) &= \sin.\tfrac{1}{2}m'\pi\,.\,s \\
&\quad + \sin.\tfrac{3}{2}(m' - n')\pi\,.\,\frac{m'}{n'}s' \\
\sin.\frac{m'}{n'}\left(\frac{3n'-1}{2}\pi + x\right) &= \sin.\tfrac{1}{2}m'\pi.s \\
&\quad + \sin.\tfrac{3}{2}(m' - n')\pi\,.\,\frac{m'}{n'}s'
\end{aligned}\right\}\ \cdots\cdots\ [10].$$

But since m' is even and n' odd,

$$\sin.\tfrac{1}{2}m'\pi = 0, \quad \sin.\tfrac{1}{2}(m' - n')\pi = \pm 1,$$
$$\sin.\tfrac{3}{2}m'\pi = 0, \quad \sin.\tfrac{3}{2}(m' - n')\pi = \pm 1.$$

Hence

$$\left.\begin{aligned} \sin.\frac{m'}{n'}\left(\frac{n'-1}{2}\pi + x\right) &= \pm\frac{m'}{n'}s' \\ \sin.\frac{m'}{n'}\left(\frac{3n'+1}{2}\pi - x\right) &= \pm\frac{m'}{n'}s' \end{aligned}\right\} \cdots\cdots [11].$$

$$\left.\begin{aligned} \sin.\frac{m'}{n'}\left(\frac{n'+1}{2}\pi - x\right) &= \pm\frac{m'}{n'}s' \\ \sin.\frac{m'}{n'}\left(\frac{3n'-1}{2}\pi + x\right) &= \pm\frac{m'}{n'}s' \end{aligned}\right\} \cdots\cdots [12].$$

The first two being true when n' is of the form $4t + 1$, and the last when of the form $4t - 1$. The sign $+$ is used when $\frac{1}{2}(m' - n' + 1)$ is odd, and $-$ when even.

(385.) If m' be odd,

$$\sin.\tfrac{1}{2}m'\pi = \pm 1, \quad \sin.\tfrac{1}{2}(m' - n')\pi = 0,$$
$$\sin.\tfrac{3}{2}m'\pi = \pm 1, \quad \sin.\tfrac{3}{2}(m' - n')\pi = 0.$$

Hence the formulæ [10] become

$$\left.\begin{aligned} \sin.\frac{m'}{n'}\left(\frac{n'-1}{2}\pi + x\right) &= \pm s \\ \sin.\frac{m'}{n'}\left(\frac{3n'+1}{2}\pi - x\right) &= \pm s \\ \sin.\frac{m'}{n'}\left(\frac{n'+1}{2}\pi - x\right) &= \pm s \\ \sin.\frac{m'}{n'}\left(\frac{3n'-1}{2}\pi + x\right) &= \pm s \end{aligned}\right\} \cdots\cdots [13],$$

the sign $+$ being used when $\frac{1}{2}(m' + 1)$ is odd, and $-$ when even.

(386.) The series s and s' in [9] being the same as those in [3], their law and properties when m is an integer have been already determined.

It is obvious that when m is even, we have

			recip. of 10.
	22.	sec.(	
2.†			recip. of 1.
2.	23.	cosec	
2.			recip. of 2.
	24.	cosec	
			3.
2.	25.*	sin.ω	4.
	26.*	sin.ω	11.
	27.*	cos.ω	12.
	28.*	cos.ω	
			$25 \div 26$.
	29.*	$\frac{\text{sin.}\omega}{\text{sin.}\omega}$	
			$25 \div 27$.
	30.*	$\frac{\text{sin.}\omega}{\text{cos.}}$	
10.			
10.			$25 \div 28$.
	31.	$\frac{\text{sin.}}{\text{cos.}}$	
10.			
			$26 \div 27$.
	32.*	$\frac{\text{sin.}}{\text{cos.}}$	
10.			
			$26 \div 28$.
	33.	$\frac{\text{sin.}}{\text{cos.}}$	
			$27 \div 28$.
	34.	$\frac{\text{cos.}}{\text{cos.}}$	
			1. $(\omega = \omega')$
	35.*	sin	11, 13. $(\omega = \omega')$.
9.	36.*	co	
			17. $(\omega = \omega')$.
10.	37.*	ta	
			19. $(\omega = \omega')$.
1.	38.	cot	
			21. $(\omega = \omega')$.
2.	39.	sec	
			23. $(\omega = \omega')$.
f 9.	40.	cos	
			36.
	41.*	si	

42.* cos.$\frac{1}{2}$

43.* sin.

44. tan.$\frac{1}{2}$

45. cot.$\frac{1}{2}$

46. sec.$\frac{1}{2}$

47. cosec.$\frac{1}{2}$

48.* sin.45

49.* cos.60

50.* sin.30

51. $\sqrt{2}$sin.

52. sin.

53. sin.

54. tan

55. cot.

56. sec.

57. $\frac{\tan}{\tan}$

58. tan

59. cot.

Rc

Sid

GIVEN.			SOUGHT.			
;	b	A	B	C	c	$\sin.\mathrm{B} = \frac{b}{a}\sin.\mathrm{A}, \quad \mathrm{C} = 180^0 - \mathrm{A} - \mathrm{B}, \quad c =$
!	b	C	A	B	c	$\tan.\tfrac{1}{2}(\mathrm{A}-\mathrm{B}) = \frac{a-b}{a+b}\cot.\tfrac{1}{2}\mathrm{C}, \quad \cos.\tfrac{1}{2}(\mathrm{A}-\mathrm{B}) = \frac{a+b}{c}\sin.$
						Let $\sin.^2\theta = \frac{4ab\cos.^2\frac{1}{2}\mathrm{C}}{(a+b)^2}, \quad \because c = (a+b)c($
.	B	a	b	c	C	$b = a\frac{\sin.\mathrm{B}}{\sin.\mathrm{A}}, \quad c = a\frac{\sin.(\mathrm{A}+\mathrm{B})}{\sin.\mathrm{A}}, \quad \mathrm{C} = 180^\circ$ -
.	B	c	a	b	C	$a = c\frac{\sin.\mathrm{A}}{\sin.(\mathrm{A}+\mathrm{B})}, \quad b = c\frac{\sin.\mathrm{B}}{\sin.(\mathrm{A}+\mathrm{B})}, \quad \mathrm{C} =$
!	b	c	A	B	C	$\sin.^2\tfrac{1}{2}\mathrm{A} = \frac{(s-b)(s-c)}{bc}, \quad \cos.^2\tfrac{1}{2}\mathrm{A} = \frac{s(s-a)}{bc},$
						$\sin.^2\tfrac{1}{2}\mathrm{B} = \frac{(s-a)(s-c)}{ac}, \quad \cos.^2\tfrac{1}{2}\mathrm{B} = \frac{s(s-b)}{ac},$
						$\sin.^2\tfrac{1}{2}\mathrm{C} = \frac{(s-a)(s-b)}{ab}, \quad \cos.^2\tfrac{1}{2}\mathrm{C} = \frac{s(s-c)}{ab},$

I.

Values for the side c.

1. $a\dfrac{\sin.C}{\sin.A}$.
2. $b\dfrac{\sin.C}{\sin.B}$.
3. $a\cos.B + b\cos.A$.
4. $(a^2 + b^2 - 2ab\cos.C)$
5. $[(a + b)^2 - 4ab\cos.^2$
6. $[(a - b)^2 + 4ab\sin.^2$
7. $b\cos.A \pm a(1 - \sin.^2$
8. $b\cos.A \pm (a^2 - b^2\sin.$
9. $a\cos.B \pm b(1 - \sin.^2$
10. $a\cos.B \pm (b^2 - a^2\sin$
11. $\dfrac{a}{\cos.B + \sin.B\cot.C}$.
12. $\dfrac{b}{\cos.A + \sin.A\cot.C}$.
13. $a\cos.B + a\sin.B\cot.A$
14. $b\cos.A + b\sin.A\cot.$

II.

Values of sin.C.

1. $\sin.(A + B)$.
2. $\sin.A\cos.B + \sin B\cos.$
3. $\dfrac{c}{b}\sin.B$.

Resolution of

Sides *a*, *b*,

GIVEN.		SOUGHT.				
c	B	*b*	*a*	A	$\sin.b = \sin.c \sin.B$	ta
c	A	*a*	*b*	B	$\sin.a = \sin.c \sin.A$	ta
c	*b*	B	*a*	A	$\sin.B = \dfrac{\sin.b}{\sin.c}$	co
c	*a*	A	*b*	B	$\sin.A = \dfrac{\sin.a}{\sin.c}$	co
a	*b*	*c*	A	B	$\cos.c = \cos.a \cos.b$	ta
a	A	*c*	*b*	B	$\sin.c = \dfrac{\sin.a}{\sin.A}$	si
b	B	*c*	*a*	A	$\sin.c = \dfrac{\sin.b}{\sin.B}$	si
a	B	*c*	*b*	A	$\tan.c = \dfrac{\tan.a}{\cos.B}$	ta
b	A	*c*	*a*	B	$\tan.c = \dfrac{\tan.b}{\cos.A}$	ta
A	B	*c*	*a*	*b*	$\cos.c = \cot.A \cot.B$	co

F

1. $\cos.\left|\begin{matrix} a \\ b \\ c \end{matrix}\right. - \cos.$

2. $\cos.(b \pm c)$

3. $\sin.b \sin.c$ co

4. $\sin.b \sin.c$ si

5. $\sin.b \sin.c$ si

6. $\tan.^2 \tfrac{1}{2}A =$

7. $\dfrac{\sin.A}{\sin.a} = \dfrac{\sin.B}{\sin.b}$

8. $\cos.\left|\begin{matrix} A \\ B \\ C \end{matrix}\right. \cos.\left|\begin{matrix} c \\ a \\ b \end{matrix}\right.$

9. $\sin.\left|\begin{matrix} a \\ b \\ c \end{matrix}\right. \cos.\left|\begin{matrix} B \\ C \\ A \end{matrix}\right.$

10. $\sin.\left|\begin{matrix} a \\ b \\ c \end{matrix}\right. \cos.\left|\begin{matrix} C \\ A \\ B \end{matrix}\right.$

11. $\sin.(a + b$

12. $\sin.(a - b$

13. $\sin.\tfrac{1}{2}(A +$

14. $\sin.\tfrac{1}{2}(A -$

15. $\cos.\tfrac{1}{2}(A +$

16. $\cos.\tfrac{1}{2}(A -$

T

17. $\tan.\frac{1}{2}(\mathrm{A} + \mathrm{B}) = \dfrac{\cos.\frac{1}{2}(a-b)}{\cos.\frac{1}{2}(a+b)} \cot.\frac{1}{2}\mathrm{C}.$

18. $\tan.\frac{1}{2}(\mathrm{A} - \mathrm{B}) = \dfrac{\sin.\frac{1}{2}(a-b)}{\sin.\frac{1}{2}(a+b)} \cot.\frac{1}{2}\mathrm{C}.$

19. $\cos.\left|\begin{matrix}\mathrm{A}\\ \mathrm{B}\\ \mathrm{C}\end{matrix}\right. - \cos.\left|\begin{matrix}a\\ b\\ c\end{matrix}\right. \sin.\left|\begin{matrix}\mathrm{B}\\ \mathrm{C}\\ \mathrm{A}\end{matrix}\right. \sin.\left|\begin{matrix}\mathrm{C}\\ \mathrm{A}\\ \mathrm{B}\end{matrix}\right. + \cos.\left|\begin{matrix}\mathrm{B}\\ \mathrm{C}\\ \mathrm{A}\end{matrix}\right. \cos.\left|\right.$

20. $\sin.\mathrm{B} \sin.\mathrm{C} \cos.^2\frac{1}{2}a = \cos.(\mathrm{S} - \mathrm{B}) \cos.(\mathrm{S} -$

21. $\sin.\mathrm{B} \sin.\mathrm{C} \sin.^2\frac{1}{2}a = -\cos.\mathrm{S} \cos.(\mathrm{S} - \mathrm{A}$

22. $\sin.\mathrm{B} \sin.\mathrm{C} \sin.a = 2\mathrm{N},$

23. $\cot.^2\frac{1}{2}a = -\dfrac{\cos.(\mathrm{S}-\mathrm{B}) \cos.(\mathrm{S}-\mathrm{C})}{\cos.\mathrm{S} \cos.(\mathrm{S}-\mathrm{A})}.$

24. $\dfrac{\sin.a}{\sin.\mathrm{A}} = \dfrac{\sin.b}{\sin.\mathrm{B}} = \dfrac{\sin.c}{\sin.\mathrm{C}} = \dfrac{2\mathrm{N}}{\sin\mathrm{A} \sin.\mathrm{B} \sin.\mathrm{C}}.$

25. $4\mathrm{N}n = \sin.a \sin.b \sin.c \sin.\mathrm{A} \sin.\mathrm{B} \sin.\mathrm{C}.$

26. $\dfrac{\sin.a}{\sin.\mathrm{A}} = \dfrac{\sin.b}{\sin.\mathrm{B}} = \dfrac{\sin.c}{\sin.\mathrm{C}} = \dfrac{n}{\mathrm{N}} = \dfrac{2\mathrm{N}}{\sin.\mathrm{A} \sin.\mathrm{B}\ \mathrm{si}}$

27. $\cos.\left|\begin{matrix}a\\ b\\ c\end{matrix}\right. \cos.\left|\begin{matrix}\mathrm{C}\\ \mathrm{A}\\ \mathrm{B}\end{matrix}\right. - \sin.\left|\begin{matrix}a\\ b\\ c\end{matrix}\right. \cot.\left|\begin{matrix}b\\ c\\ a\end{matrix}\right. + \sin.\left|\begin{matrix}\mathrm{C}\\ \mathrm{A}\\ \mathrm{B}\end{matrix}\right. \cot$

28. $\sin.\left|\begin{matrix}\mathrm{A}\\ \mathrm{B}\\ \mathrm{C}\end{matrix}\right. \cos.\left|\begin{matrix}b\\ c\\ a\end{matrix}\right. - \cos.\left|\begin{matrix}\mathrm{B}\\ \mathrm{C}\\ \mathrm{A}\end{matrix}\right. \sin.\left|\begin{matrix}\mathrm{C}\\ \mathrm{A}\\ \mathrm{B}\end{matrix}\right. - \sin.\left|\begin{matrix}\mathrm{B}\\ \mathrm{C}\\ \mathrm{A}\end{matrix}\right. \cos.$

29. $\sin.\left|\begin{matrix}\mathrm{A}\\ \mathrm{B}\\ \mathrm{C}\end{matrix}\right. \cos.\left|\begin{matrix}c\\ a\\ b\end{matrix}\right. - \cos.\left|\begin{matrix}\mathrm{C}\\ \mathrm{A}\\ \mathrm{B}\end{matrix}\right. \sin.\left|\begin{matrix}\mathrm{B}\\ \mathrm{C}\\ \mathrm{A}\end{matrix}\right. - \sin.\left|\begin{matrix}\mathrm{C}\\ \mathrm{A}\\ \mathrm{B}\end{matrix}\right. \cos$

30. $\sin.(\mathrm{A} + \mathrm{B}) \cos.^2\frac{1}{2}c - \cos.\frac{1}{2}(a - b)\ \mathrm{co}$

31. $\sin.(\mathrm{A} - \mathrm{B}) \sin.^2\frac{1}{2}c - \sin.\frac{1}{2}(a - b)\ \mathrm{sir}$

32. $\sin.\frac{1}{2}(a + b) = \dfrac{\sin.\frac{1}{2}c}{\sin.\frac{1}{2}\mathrm{C}} \cos.\frac{1}{2}(\mathrm{A} - \mathrm{B}).$

33. $\sin.\frac{1}{2}(a - b) = \dfrac{\sin.\frac{1}{2}c}{\cos.\frac{1}{2}\mathrm{C}} \sin.\frac{1}{2}(\mathrm{A} - \mathrm{B}).$

34. $\cos.\frac{1}{2}(a + b) = \dfrac{\cos.\frac{1}{2}c}{\sin.\frac{1}{2}\mathrm{C}} \cos.\frac{1}{2}(\mathrm{A} + \mathrm{B}).$

35. $\cos.\frac{1}{2}(a - b) = \dfrac{\cos.\frac{1}{2}c}{\cos.\frac{1}{2}\mathrm{C}} \sin.\frac{1}{2}(\mathrm{A} + \mathrm{B}).$

36. $\tan.\frac{1}{2}(a+b) =$

37. $\tan.\frac{1}{2}(a-b) =$

38. $\frac{\sin.(a-b)}{\sin.(a+b)} =$ ta

39. $\frac{\sin.(A-B)}{\sin.(A+B)} =$ c

40. $\frac{\sin.\frac{1}{2}(a-b)\cot.\frac{1}{2}}{\cos.\frac{1}{2}(a+b)}$

41. $\sin.A \sin.B -$ si

42. $\sin.A \sin.a =$ [t

43. $\sin.\left|\begin{matrix}A\\B\\C\end{matrix}\right.\cot.\left|\begin{matrix}a\\b\\c\end{matrix}\right. =$ s

44. $\sin.\left|\begin{matrix}A\\B\\C\end{matrix}\right.\cot.\left|\begin{matrix}a\\b\\c\end{matrix}\right. =$ s

45. $\sin.\left|\begin{matrix}a\\b\\c\end{matrix}\right.\cot.\left|\begin{matrix}A\\B\\C\end{matrix}\right. =$ s

46. $\sin.\left|\begin{matrix}a\\b\\c\end{matrix}\right.\cot\left|\begin{matrix}A\\ \\C\end{matrix}\right.$ s

47. $\sin.^2 c \sin.(a - b$

48. $\cos.c = \cos.(a -$

49. $\sin.^2\frac{1}{2}c = \sin.^2(a$

50. $\cos.^2\frac{1}{2}c = \cos.^2\frac{1}{2}($

51. $\tan.^2\frac{1}{2}c = \frac{\sin.(a}{\sin.(a}$

52. $\cos.c = -\cos.($

53. $\cos.^2\frac{1}{2}c = \sin.^2\frac{1}{2}($

54. $\sin.^2\frac{1}{2}c = \cos.^2\frac{1}{2}($

55. $\cot.^2\frac{1}{2}c = \frac{\sin.(A}{\sin.(A}$

56. $\sin.\tfrac{1}{2}A \sin.\tfrac{1}{2}B \sin.\tfrac{1}{2}C = \dfrac{\sin.(s-a)\sin.(s-b)\sin.(}{\sin.a \sin.b \sin.c}$

57. $\cos.\tfrac{1}{2}A \cos.\tfrac{1}{2}B \cos.\tfrac{1}{2}C = \dfrac{[\sin.s \sin.s \sin.s \sin.(s-a}{\sin.a \sin}$

58. $\tan.\tfrac{1}{2}A \tan.\tfrac{1}{2}B \tan.\tfrac{1}{2}C = \Big[\dfrac{\sin.(s-a)\sin.(s-b)\sin}{\sin.s \sin.s \sin.s}$

59. $\cos.\tfrac{1}{2}a \cos.\tfrac{1}{2}b \cos.\tfrac{1}{2}c = \dfrac{\cos.(S-A)\cos.(S-B)\cos}{\sin.A \sin.B \sin.C}$

60. $\sin.\tfrac{1}{2}a \sin.\tfrac{1}{2}b \sin.\tfrac{1}{2}c = \dfrac{[-\cos.S \cos.S \cos.S \cos.(}{\sin.A}$

61. $\cot.\tfrac{1}{2}a \cot.\tfrac{1}{2}b \cot.\tfrac{1}{2}c = \Big[-\dfrac{\cos.(S-A)\cos.(S-B)}{\sin.A \sin.B \sin}$

62. $n = \tfrac{1}{2}[\sin.^2 a \sin.^2 b \sin.^2 c \sin.A \sin.B \sin.C]^{\frac{1}{3}}.$

63. $N = \tfrac{1}{2}[\sin.a \sin.b \sin.c \sin.^2 A \sin.^2 B \sin.^2 C]^{\frac{1}{3}}.$

64. $\dfrac{n}{N} = \left[\dfrac{\sin.a \sin.b \sin.c}{\sin.A \sin.B \sin.C}\right]^{\frac{1}{3}}.$

65. $.s = \dfrac{n^2}{\sin.a \sin.b \sin.c \sin.\tfrac{1}{2}A \sin.\tfrac{1}{2}B \sin.\tfrac{1}{2}C} = 2\sin$

66. $\sin.s = \dfrac{\sin.a \sin.b \sin.c \cos.\tfrac{1}{2}A \cos.\tfrac{1}{2}B \cos.\tfrac{1}{2}C}{n} = 2$

67. $\sin.^2 s = \dfrac{n}{\tan.\tfrac{1}{2}A \tan.\tfrac{1}{2}B \tan.\tfrac{1}{2}C}$ . . .

68. $\cos.S = -\dfrac{N^2}{\sin.A \sin.B \sin.C \cos.\tfrac{1}{2}a \cos.\tfrac{1}{2}b \cos.\tfrac{1}{2}c} =$

69. $\cos.S = \dfrac{\sin.A \sin.B \sin.C \sin.\tfrac{1}{2}a \sin.\tfrac{1}{2}b \sin.\tfrac{1}{2}c}{N} = 2$

70. $\cos.^2 S = \dfrac{N}{\cot.\tfrac{1}{2}a \cot.\tfrac{1}{2}b \cot.\tfrac{1}{2}c}.$ . . .

71. $n^2 = 4\cos.^2\tfrac{1}{2}a \cos.^2\tfrac{1}{2}b \cos.^2\tfrac{1}{2}c - [1 - \cos.^2\tfrac{1}{2}a -$

72. $n^2 = 4\cos.^2\tfrac{1}{2}a \sin.^2\tfrac{1}{2}b \sin.^2\tfrac{1}{2}c - [1 + \cos.^2\tfrac{1}{2}a -$

73. $N^2 = 4\sin.^2$

74. $N^2 = 4\sin.^2$

75. $\cos.s = \frac{1 -}{}$

76. $\sin.s = \frac{1 -}{}$

77. $\tan.s = \frac{}{1}$

78. $\tan.s = \frac{1}{}$

79. $\sin.(s -$

80. $\cos.(s -$

81. $\tan.(s -$

82. $\cos.(s -$

83. $\sin.(s -$

84. $\cot.(s -$

Resolution o

Sides a,

1. $\sin.\frac{1}{2}A = \left(\frac{\sin.(s-b)\sin.(s-c)}{\sin.b\sin.c}\right)^{\frac{1}{2}}$

2. $\cos.\frac{1}{2}A = \left(\frac{\sin.s\sin.(s-a)}{\sin.b\sin.c}\right)^{\frac{1}{2}}$

3. $\tan.\frac{1}{2}A = \left(\frac{\sin.(s-b)\sin.(s-c)}{\sin.s\sin.(s-a)}\right)^{\frac{1}{2}}$

4. $\sin.A =$ 2[sin.s si

5. $\cos.A =$ 2sin.½(θ - ... si

6. $\cos.A =$ cos.(a ... cos.θ sin.

Give

1. $\sin.\frac{1}{2}a = \left(\frac{-\cos.S\cos.(S-A)}{\sin.B\sin.C}\right)^{\frac{1}{2}}$

2. $\cos.\frac{1}{2}a = \left(\frac{\cos.(S-B)\cos.(S-C)}{\sin.B\sin.C}\right)^{\frac{1}{2}}$

3. $\cot.\frac{1}{2}a = \left(-\frac{\cos.(S-B)\cos.(S-C)}{\cos.S\cos.(S-A)}\right)^{\frac{1}{2}}$

4. $\sin.a =$ 2[− cos.

5. $\cos.a =$ 2cos.½(A ... si

6. $\cos.a =$ cos.(A- ... sin.B sin.

Given two sides (a

1°. To determine A and B.

1. $\sin.\frac{1}{2}(A+B) = \frac{\cos.\frac{1}{2}C}{\cos.\frac{1}{2}c}\cos.\frac{1}{2}(a-b)$

$\sin.\frac{1}{2}(A-B) = \frac{\cos.\frac{1}{2}C}{\sin.\frac{1}{2}c}\sin.\frac{1}{2}(a-b)$

2. $\cos.\frac{1}{2}(A+$

$\cos.\frac{1}{2}(A-$

---

* $\cos.^{-1}$ signifies "the angle whose cosine is." Thus $\cos.^{-1}$(c
similar signification.

73. $N^2 = 4\sin.^2$

74. $N^2 = 4\sin.^2$

75. $\cos.s = \frac{1-}{}$

76. $\sin.s = \frac{1}{}$

77. $\tan.s = \frac{}{1}$

78. $\tan.s = \frac{1}{}$

79. $\sin.(s -$

80. $\cos.(s -$

81. $\tan.(s -$

82. $\cos.(s -$

83. $\sin.(s -$

84. $\cot.(s -$

Resolution o

Sides *a*,

1. $\sin.\frac{1}{2}A = \left(\frac{\sin.(s-b)\sin.(s-c)}{\sin.b \sin.c}\right)^{\frac{1}{2}}$

2. $\cos.\frac{1}{2}A = \left(\frac{\sin.s \sin.(s-a)}{\sin.b \sin.c}\right)^{\frac{1}{2}}$

3. $\tan.\frac{1}{2}A = \left(\frac{\sin.(s-b)\sin.(s-c)}{\sin.s \sin.(s-a)}\right)^{\frac{1}{2}}$

4. $\sin.A = \frac{2[\sin.s \text{ si}}{}$

5. $\cos.A = \frac{2\sin.\frac{1}{2}(\theta -}{\text{si}}$

6. $\cos.A = \frac{\cos.(a}{\cos.\theta \sin.}$

Giv

1. $\sin.\frac{1}{2}a = \left(\frac{-\cos.S \cos.(S-A)}{\sin.B \sin.C}\right)^{\frac{1}{2}}$

2. $\cos.\frac{1}{2}a = \left(\frac{\cos.(S-B)\cos.(S-C)}{\sin.B \sin.C}\right)^{\frac{1}{2}}$

3. $\cot.\frac{1}{2}a = \left(-\frac{\cos.(S-B)\cos.(S-C)}{\cos.S \cos.(S-A)}\right)^{\frac{1}{2}}$

4. $\sin.a = \frac{2[-\cos.}{}$

5. $\cos.a = \frac{2\cos.\frac{1}{2}(A}{\text{si}}$

6. $\cos.a = \frac{\cos.(A -}{\sin.B \sin.}$

Given two sides (a

1°. To determine A and B.

1. $$\left.\begin{aligned} \sin.\tfrac{1}{2}(A+B) &= \frac{\cos.\frac{1}{2}C}{\cos.\frac{1}{2}c}\cos.\tfrac{1}{2}(a-b) \\ \sin.\tfrac{1}{2}(A-B) &= \frac{\cos.\frac{1}{2}C}{\sin.\frac{1}{2}c}\sin.\tfrac{1}{2}(a-b) \end{aligned}\right\}$$

2. $\cos.\frac{1}{2}(A +$

$\cos.\frac{1}{2}(A -$

---

* $\cos.^{-1}$ signifies "the angle whose cosine is." Thus $\cos.^{-1}$(c
similar signification.

$$3.\ \tan.\tfrac{1}{2}(A+B)=\frac{\cos.\frac{1}{2}(a-b)}{\cos.\frac{1}{2}(a+b)}\cot.\tfrac{1}{2}C \qquad \tan.\tfrac{1}{2}(A-B)=\frac{\sin.\frac{1}{2}(a-b)}{\sin.\frac{1}{2}(a+b)}\cot.\tfrac{1}{2}C$$

2°. To determine the side c.

$$1.\ \sin.c=\frac{\sin.a}{\sin.A}\sin.C.$$

$$2.\ \cos.c=\frac{\cos.a\cos.(\ldots b)}{\ldots}\quad[\theta=\tan.^{-1}(\tan\ldots$$

$$3.\ \ldots(a\ldots\ \frac{\ldots{}^{2}\tfrac{1}{2}C\sin.a\sin\ldots}{\cos.(a-b)}$$

$$4.\ \cos.c=c\ldots\ \frac{\ldots{}^{2}\tfrac{1}{2}C\sin.a\sin\ldots}{\cos.(a+b)}$$

...BSERVATIONS.

...he first formula is ...useful when A has ...previously com-...d. The quantities ...h are represented by ...re easily computed. ...y are all of the form ...N. To compute ...N, let $\theta=\tan.^{-1}\sqrt{N}$ ...$n.^2\theta = N$ ∵ $1 + N$ ...$ec.^2\theta$. To compute ...N, when $N<1$, let ...$os.^{-1}\sqrt{N}$ ∵ $1-N$ ...$n.^2\theta$. When $N>1$, $\theta=\sec.^{-1}\sqrt{N}$, ∵ $N=-\tan.^2\theta$ (237.)

1°. To determine a and b.

$$1.\ \sin.\tfrac{1}{2}(a+b)=\frac{\sin.\frac{1}{2}c}{\sin.\frac{1}{2}C}\cos.\tfrac{1}{2}(A-B) \qquad \sin.\tfrac{1}{2}(a-b)=\frac{\sin.\frac{1}{2}c}{\cos.\frac{1}{2}C}\sin.\tfrac{1}{2}(A-B)$$

$$2.\ \cos.\tfrac{1}{2}(a+b)=\frac{\cos.\frac{1}{2}c}{\sin.\frac{1}{2}C}\cos.\tfrac{1}{2}(A+B) \qquad \cos.\tfrac{1}{2}(a-b)=\frac{\cos.\frac{1}{2}c}{\cos.\frac{1}{2}C}\sin.\tfrac{1}{2}(A+B)$$

...hese formulæ are ...ogous to those in ...ast case, and similar ...rvations are appli-...e.

2°. To determine the angle C.

$$1.\ \sin.C=\frac{\sin.A}{\sin.a}\sin.c.$$

$$2.\ \cos.C=\frac{\cos.A\cos.(\theta+B)}{\sin.\theta}\quad[\theta=\tan.^{-1}\ldots$$

5. $\cos.\frac{1}{2}c = -\sin.\frac{1}{2}(A-B)\sin.\frac{1}{2}c\left[1+\frac{\sin.^2\frac{1}{2}(A+B)}{\sin.^2\frac{1}{2}(A-B)}\cot.^2\frac{1}{2}c\right]^{\frac{1}{2}} = -\sin.\frac{1}{2}$

6. $\sin.\frac{1}{2}c = \cos.\frac{1}{2}(A-B)\sin.\frac{1}{2}c\left[1+\frac{\cos.^2\frac{1}{2}(A+B)}{\cos.^2\frac{1}{2}(A-B)}\cot.^2\frac{1}{2}c\right]^{\frac{1}{2}} = \cos.\frac{1}{2}(A -$

Given two side

1°. To determine the angle B opposed to th

$\sin. B = \frac{\sin}{\sin}$

1. If $m < \sin.A$, there is no solution.
2. If $m = \sin.A$, $B = 90°$, when a and A

When a and

3. If $m > \sin.A$ and < 1, there are two s

When a and

4. If m be not < 1, that value of B only

2°. To determine the angle C.

$\cos.\varphi = \tan.b$

1. If $m > 1$, $C = \varphi$

$C = \varphi$

2. If $m < 1$ and $>$ si

3°. To determine the side c.

$\cos.\varphi = \cos.$

1. If $m > 1$, $c = \varphi$

$c = \varphi$

2. If $m < 1$, $c = \varphi$

If B be previously determined, the angle C

$\cot.\frac{1}{2}$

$\tan.\frac{1}{2}$

1°. To determine the si ATIONS.

observations
as in the last

1 If $m <$ sin.a, there
2. If $m =$ sin.a, $b =$

3. If $m >$ sin.a and

4. If m be not < 1, t

2°. To determine the sid

1. If $m > 1$, θ and φ
species.

2. If $m < 1$, φ has tw

3°. To determine the an

If $m > 1$, φ and θ are

If $m < 1$, φ has two s

If $r =$ radius of the sph

formulæ the
all expressed
to the radius
The method
the formulæ
ding ones re-
other radius
explained in

1. $\text{D} = r^2(2\text{S} - \pi)$,

$\text{D} = 2\text{S} - \pi$, (the

2. $\sin.\frac{1}{2}\text{D} = \frac{\sqrt{\sin.s \sin}}{2}$

T

3. $\cos.\frac{1}{2}D = \dfrac{\cos.\frac{1}{2}a \cos.\frac{1}{2}b + \sin.\frac{1}{2}a \sin.\frac{1}{2}b \cos}{\cos.\frac{1}{2}c}$

4. $\cos.\frac{1}{2}D = \dfrac{\cos.a + \cos.b + \cos.c + 1}{4\cos.\frac{1}{2}a \cos.\frac{1}{2}b \cos.\frac{1}{2}c} = \dfrac{\cos.^2\frac{1}{2}}{2}$

5. $\tan.\frac{1}{2}D = \sqrt{\tan.\frac{1}{2}s \tan.\frac{1}{2}(s-a) \tan.\frac{1}{2}(s -}$

6. $\tan.\frac{1}{2}D = \dfrac{\tan.\frac{1}{2}a \tan.\frac{1}{2}b \sin.C}{1 + \tan.\frac{1}{2}a \tan.\frac{1}{2}b \cos.C}$.

7. $\tan.\frac{1}{2}D = \dfrac{\sin.\frac{1}{2}a \sin.\frac{1}{2}b \sin.C}{4\cos.\frac{1}{2}(\varphi+\theta) \cos.\frac{1}{2}(\varphi-\theta)}$, $[\theta$ =

$$\sin.mx = \pm ms',$$

$+$ being used when $\frac{1}{2}m$ is even, and $-$ when odd.

And when m is odd

$$\sin.\,mx = \pm\,s,$$

$+$ being used when $\frac{1}{2}(m+1)$ is odd, and $-$ when even.

(387.) Another form for the development of $\sin.mx$ in ascending powers of the $\cos.x$ may be established by differentiating the series found for $\cos.mx$ in (372). By this process we obtain

$$m\sin.m(2n\pi \pm x) = -\cos.\tfrac{1}{2}m(4n \pm 1)\,\pi.\frac{ds}{dx}$$

$$-\cos.\tfrac{1}{2}(m-1)(4n \pm 1)\pi\,.\,m\frac{ds'}{dx},$$

$$\frac{ds}{dy} = -\frac{m^2}{1}y + \frac{m^2(m^2-2^2)}{1.2.3}y^3 - \frac{m^2(m^2-2^2)(m^2-4^2)}{1.2.3.4.5}y^5$$
$$+\cdots\cdots = -m\text{R},$$

$$\frac{ds'}{dy} = 1 - \frac{m^2-1^2}{1.2}y^2 + \frac{(m^2-1^2)(m^2-3^2)}{1.2.3.4}y^4 - \cdots\cdots = \text{R}',$$

$$\frac{dy}{dx} = -\sin.x,$$

$$\therefore \frac{ds}{dx} = m\sin.x\text{R}, \qquad \frac{ds'}{dx} = -\sin.x.\text{R}',$$

$$\therefore m\sin.m(2n\pi \pm x) = -\sin.x[\cos.\tfrac{1}{2}m(4n \pm 1)\pi.m\text{R}$$
$$-\cos.\tfrac{1}{2}(m-1)(4n \pm 1)\pi.\text{R}'] \cdots\cdots [13].$$

This being deduced directly from the formula [3] is liable to the various modifications which have been shown to be incident to [3], on assigning particular values to m and n. The several modifications of [13] which correspond to these may be deduced by differentiating the several series [5], [7], [8], &c. &c.

(388.) The laws of the series R and R$'$ are easily defined. Let T and T$'$ be their rth terms respectively

$$\text{T} = \pm\frac{m^2(m^2-2^2)(m^2-4^2)\cdots\cdots[m^2-(2r-2)^2]}{1.2.3\ldots\ldots 2r-1}y^{2r-1},$$

$$T' = \pm \frac{(m^2-1^2)(m^2-3^2)\cdots\cdots[m^2-(2r-3)^2]}{1.2.3\cdots\cdots 2(r-1)}y^{2(r-1)}.$$

The number of terms in R is only finite when m is an even integer, and in R$'$ when it is an odd integer. The number in R is evidently one less than in S when it is finite, and is therefore equal to $\frac{m}{2}$. But the number in R$'$ when it is finite is the same as in S$'$, and is therefore $\frac{m+1}{2}$.

The last terms of R and R$'$ in these cases may be obtained by differentiating those of S and S$'$, and dividing the one by $m\sin.x$, and the other by $-\sin.x$.

PROP. CI.

(389.) *To develop the cosine or sine of a multiple arc in ascending powers of the sine of the simple arc.*

Let $y = \sin.x$,

$$z = \sqrt{1-y^2} + y\sqrt{-1},$$
$$z^m = (\sqrt{1-y^2} + y\sqrt{-1})^m;$$

and since

$$2\cos.mx = z^m + z^{-m},$$
$$2\sqrt{-1}\sin.mx = z^m - z^{-m},$$

the problem will be solved by obtaining the development of z^m in ascending powers of y.

Let

$$z^m = A_0 + A_1 y + A_2 y^2 + \cdots\cdots$$

By proceeding exactly as in (371.), we shall obtain

$$z^m = A_0\left\{1 - \frac{m^2}{1.2}y^2 + \frac{m^2(m^2-2^2)}{1.2.3.4}y^4 - \frac{m^2(m^2-2^2)(m^2-4^2)}{1.2.3.4.5.6.}y^6 + \cdots\cdots\right\}$$

$$+ A_1\left\{y - \frac{m^2-1^2}{1.2.3}y^3 + \frac{(m^2-1^2)(m^2-3^2)}{1.2.3.4.5}y^5 - \dots\right\}.$$

The values of A_0 and A_1 may be determined by making, as in (371.), $y = 0$ in the two values of z^m and $\frac{d(z^m)}{dy}$, and equating the results, which gives *

$$A_0 = (1)^{\frac{m}{2}}, \quad A_1 = m\sqrt{-1}\,(1)^{\frac{m-1}{2}}.$$

The value of z^{-m} may be deduced from that of z^m by changing the sign of m. Hence, if the series which enter these values be Q, Q', we obtain

$$z^m = (1)^{\frac{m}{2}}Q + \sqrt{-1}(1)^{\frac{m-1}{2}}mQ',$$

$$z^{-m} = (1)^{-\frac{m}{2}}Q - \sqrt{-1}(1)^{-\frac{m-1}{2}}mQ',$$

$$\therefore 2\cos.mx = [(1)^{\frac{m}{2}} + (1)^{-\frac{m}{2}}]Q + \sqrt{-1}[(1)^{\frac{m-1}{2}} + (1)^{-\frac{m-1}{2}}]mQ',$$

$$2\sqrt{-1}\sin.mx = [(1)^{\frac{m}{2}} - (1)^{-\frac{m}{2}}]Q + \sqrt{-1}[(1)^{\frac{m-1}{2}} + (1)^{-\frac{m-1}{2}}]mQ'.$$

It will be observed, that by changing x into $2n\pi + x$, no change is made on the series Q and Q'; but there is a change made upon the first member of each equation. The coefficients of Q and Q' have exactly as many different values as the first members of the equation. This is a circumstance which has been hitherto overlooked †.

The above formulæ can be cleared of imaginary quantities by the usual method,

$$(1)^{\frac{m}{2}} = \cos.nm\pi + \sqrt{-1}\sin.nm\pi,$$

$$(1)^{\frac{m-1}{2}} = \cos.n(m-1)\pi + \sqrt{-1}\sin.n(m-1)\pi,$$

---

* Lagrange, and all mathematicians after him, have fallen into an error in the determination of these coefficients. *Poinsot* has lately corrected it.

† *Poinsot*, 1825.

the number n being an indeterminate integer. All the arcs which have the same sine may be included under the formula $n\pi \pm x$, x being taken with the sign + when n is even, and − when n is odd. Hence the formulæ become

$$\cos.m(n\pi \pm x) = \cos.nm\pi.\text{Q} - \sin.n(m-1)\pi.m\text{Q}' \cdots\cdots [14],$$
$$\sin.m(n\pi \pm x) = \sin.nm\pi.\text{Q} + \cos.n(m-1)\pi.m\text{Q}' \cdots\cdots [15].$$

(390.) There are certain values of n for which each of the coefficients of these formulæ $= 0$. To determine these, let $m = \frac{m'}{n'}$, and let it be remembered that no value is supposed to be assigned to n greater than n'. We have thence the following conditions:

$$\cos.nm\pi = 0, \quad \because n = \frac{n'}{2}, \text{ or } n = \frac{3n'}{2},$$
$$\cos.n(m-1)\pi = 0, \quad \because n = \frac{n'}{2}, \text{ or } n = \frac{3n'}{2},$$
$$\sin.nm\pi = 0, \quad \because n = 0, \text{ or } n = n',$$
$$\sin.n(m-1)\pi = 0, \quad \because n = 0, \text{ or } n = n'.$$

The first two conditions can only be satisfied when the denominator (n') of m is even. Hence it follows, that of all the arcs whose sines have any given value, there are always two (x) for which the formulæ [14], [15], are reduced to a single series. These two arcs are of the forms $\frac{n'}{2}\pi + x$, $\frac{3n'}{2}\pi - x$, or $\frac{n'}{2}\pi - x$, $\frac{3n'}{2} + x$. For these two values of n we have

$$\cos.m\text{x} = \pm m\text{Q}', \quad \sin.m\text{x} = \pm \text{Q}.$$

The last two conditions can be fulfilled, whatever be the value of n', and the formulæ [14], [15], become

$$\cos.m\text{x} = \pm \text{Q}, \quad \sin.m\text{x} = \pm m\text{Q}';$$

where x is an arc of the form x or $n'\pi + x$ when n' is even, and x or $n'\pi - x$ when n' is odd.

It appears, therefore, that among the values of an arc,

whose sine is given, there are always two, the cosines and sines of whose multiples admit of being expressed by a single series. In this respect, the developments by the powers of the sine differ from those by the powers of the cosine, in which, when the denominator of m is even, there is no value of the simple arc, the cosine or sine of whose multiple can be developed in a single series.

(391.) If m be an integer, one of the coefficients of each of the formulæ [14], [15], must necessarily $= 0$.

This comes within the case in which m has an odd denominator, since the denominator is unity, and since no value is supposed to be given to n greater than n', it is in this case necessarily $= 1$. Hence in this case

$$\cos.mx = \pm Q, \quad \sin.mx = \pm mQ'.$$

The double sign applies to the two values of x, scil. x and $\pi - x$, which have the same sine. The value of $\cos.mx$ with the sign $+$ is used when m is even, and that with the sign $-$ when m is odd; and in the value of $\sin.mx$ the sign $+$ is used when m is odd, and $-$ when m is even.

When m is even, the series Q is finite and Q' infinite, and when m is odd, Q' is finite and Q infinite. The form of these series being the same as the series S, S', in (371.) the law, the number of terms when finite, and the last term is determined in the same manner.

PROP. CII.

(392.) *To develop the sine and cosine of a multiple arc in a series of ascending powers of the tangent of the simple arc.*

By developing the formula

$$\cos.mx + \sqrt{-1}\sin.mx = (\cos.x + \sqrt{-1}\sin.x)^m;$$

by the binomial theorem we shall obtain

$$\cos.mx + \sqrt{-1}\sin.mx = R + \sqrt{-1}R' \ldots [16],$$

where R represents the sum of the odd, and R' of the even terms of the development, and therefore

$$R = \cos.^m x - A_2\cos.^{m-2}x\sin.^2 x + A_4\cos.^{m-4}x\sin.^4 x - \cdots$$
$$R' = A_1\cos.^{m-1}x\sin.x - A_3\cos.^{m-3}x\sin.^3 x + A_5\cos.^{m-5}x\sin.^5 x - \cdots$$

where A_1, A_2, A_3, ... represent the coefficients of the second and succeeding terms of the expanded binomial, whose exponent is m.

As each side of the equation [16] consists partly of real and partly of imaginary quantities, it is equivalent to two distinct equations, between each separately. If we consider R composed exclusively of real, and $\sqrt{-1}R'$ of imaginary quantities, we should therefore have

$$\left.\begin{array}{l}\cos.mx = R \\ \sin.mx = R'\end{array}\right\}[17].$$

These formulæ, which were first published by *John Bernoulli* in the Leipsic Acts, 1701, have been, even to the present day, considered as exact and general. This, however, is not the case.

To explain this, let

$$T = 1 - A_2\tan.^2 x + A_4\tan.^4 x - \cdots$$
$$T' = A_1\tan.x - A_3\tan.^3 x + A_5\tan.^5 x - \cdots$$
$$\therefore R = \cos.^m x.T,$$
$$R' = \cos.^m xT'.$$

By changing x into $2n\pi + x$, the factors T, T', of the second members of

$$\cos.mx = \cos.^m x.T,$$
$$\sin.mx = \cos.^m x.T',$$

undergo no change, since these arcs have the same tangent, and since T, T', include no powers except integral powers of $\tan.x$, they can have each but one value for an arc, whose sine and cosine are given. The first factor $\cos.^m x$ has, however, as many different values as there are units in the denominator of m, of which two, at most, can be real, and all

the others must be imaginary. On the other hand, for an arc whose sine and cosine are given, and which is of the form $2n\pi + x$, n being any integer, the first members of these equations have as many different values as there are units in the denominator of m, and *all* these values are *real*. Thus the two members of the equations are inconsistent.

It is not difficult to perceive that this absurdity has arisen from the false assumption that the real and imaginary parts of the second member of [16] were R and $\sqrt{-1}\text{R}'$. We shall find upon consideration that neither of these quantities are altogether real, or altogether imaginary, but that each of them is composed partly of real and partly of imaginary quantities, and is of the form $a + \sqrt{-1}.b$.

In the formula

$$\cos.mx + \sqrt{-1}\sin.mx = \cos.^m x(\text{T} + \sqrt{-1}\text{T}'),$$

let the absolute, real, or arithmetical value of $\cos.^m x$, $\cos.x$ being considered merely as a number, be P. It is plain that its several algebraical values will be expressed by the formula $\text{P}(\pm 1)^m$. And since

$$(\pm 1)^m = \cos.mn\pi + \sqrt{-1}\sin.mn\pi,$$

$$\therefore \cos.^m x = \text{P}(\cos.mn\pi + \sqrt{-1}\sin.mn\pi),$$

the indeterminate integer n being even when $\cos.x$ is positive, and odd when it is negative.

Making this substitution in the former equation, and in place of x, substituting the general formula $n'\pi \pm x$ for all arcs having the same cosine, in which the sign $+$ is used when n' is even, and $-$ when it is odd, we obtain

$$\begin{aligned}&\cos.m(n'\pi \pm x) + \sqrt{-1}\sin.m(n'\pi \pm x)\\ &= \text{P}(\text{T}\cos.mn\pi - \text{T}'\sin.mn\pi)\\ &+ \sqrt{-1}.\text{P}(\text{T}\sin.mn\pi + \text{T}'\cos.mn\pi).\end{aligned}$$

Here the real and imaginary parts are separated on each side, and equating them, we have

$$\cos.m(n'\pi \pm x) = \text{P}(\text{T}\cos.mn\pi - \text{T}'\sin.mn\pi),$$
$$\sin.m(n'\pi \pm x) = \text{P}(\text{T}\sin.mn\pi + \text{T}'\cos.mn\pi).$$

Each member of these equations is susceptible of as many different values as there are units in the denominator of m. But it remains still to be determined which of the values of the second members correspond or are equal to those of the first severally. In other words, it is necessary to determine what relation subsists between the indeterminate integers n' and n, neither of which are supposed to exceed twice the denominator of m. To determine this, let $x = 0$, $\therefore \text{P} = 1$, $\text{T} = 1$, $\text{T}' = 0$. Hence

$$\cos.mn'\pi = \cos.mn\pi,$$
$$\sin.mn'\pi = \sin.mn\pi,$$
$$\therefore n' = n.$$

These integers are therefore always equal, and the formulæ become

$$\cos.m(n\pi \pm x) = \text{P}(\text{T}\cos.mn\pi - \text{T}'\sin.mn\pi) \cdot\cdot\cdot\cdot [18],$$
$$\sin.m(n\pi \pm x) = \text{P}(\text{T}\sin.mn\pi + \text{T}'\cos.mn\pi) \cdot\cdot\cdot\cdot [19].$$

Whether the odd or even integers are to be substituted for n in these formulæ, and whether x is to be taken with $+$ or $-$, is to be determined by the signs of $\sin.x$ and $\cos.x$, which are supposed to be given. If $\cos.x$ be positive, the values of n are to be selected from the series

$$0,\ 2,\ 4,\ 6,\ \cdot\cdot\cdot\cdot;$$

if it be negative, they are to be selected from

$$1,\ 3,\ 5.\ \cdot\cdot\cdot\cdot$$

If $\sin.x$ be positive, x is to be taken with $+$, and if negative, with $-$. In all cases, however, the coefficient R in the second members is to be considered as an abstract number independent of any sign.

If m be an integer, the formulæ are reduced to the forms

$$\cos.mx = \cos.^m x\text{T}, \quad \sin.mx = \cos.^m x.\text{T}',$$

which have hitherto been taken to be general for all values of m.

There are, however, particular values of n' even when m is a fraction, for which one or other of the series by which $\cos.mx$ and $\sin.mx$ are expressed will disappear. In order that $\cos.mn\pi$ should $= 0$, it is necessary that mn should be a fraction whose denominator is 2, and therefore whose numerator is an odd number. This can only happen when m is a fraction with an even denominator, and therefore an odd numerator, and when n is equal to half the denominator. Also in this case, if half the denominator of m be an even number, it is necessary that $\cos.x$ should be positive (otherwise n should be odd), and if half the denominator be an odd number, it is necessary that $\cos.x$ should be negative, for otherwise n should be even. Hence we may conclude, that if m be a fraction with an even denominator, there is always one arc, whose cosine has any given positive value when half the denominator of m is even, and whose cosine has any given negative value when half the denominator is odd, which is such, that each of the formulæ [18], [19], are reduced to a single series, since under the conditions just stated,

$$\cos.mn'\pi = 0, \quad \sin.mn\pi = \pm 1.$$

In order that $\sin.mn\pi = 0$, it is necessary that mn should be an integer, and therefore that n should be equal, either to the denominator of m, or to twice the denominator. In each case $\sin.mn\pi = 0$ and $\cos.mn\pi = \pm 1$. If $\cos.x$ be positive, n must be even, and in this case, if the denominator of m be even, there are two values of n, which will reduce the formulæ [18], [19], to a single series; but if it be odd, since n must be even, there is but one value will satisfy this condition. If $\cos.x$ be negative, n must be odd, and therefore, when the denominator of m is odd, there is but one value of n, which will reduce the formulæ to a single series, and when m is even, there is no value of n will effect this.

It appears, therefore, that when m is an integer, $\cos.mx$ and $\sin.mx$ can always be expressed in a single series of

powers of the tangent; but that when m is a fraction, there are only certain values of an arc of a given sine and cosine, which admit of a development without both the series of [9], [10], and that in some cases there is no arc which admits it.

If the two formulæ [18], [19], be divided one by the other, we shall obtain

$$\tan.m(n\pi \pm x) = \frac{\text{T}\cos.mn\pi - \text{T}'\sin.mn\pi}{\text{T}\sin.mn\pi + \text{T}'\cos.mn\pi},$$

$$= \frac{\text{T} - \text{T}'\tan.mn\pi}{\text{T}\tan.mn\pi + \text{T}'} \quad . \quad . \quad . \quad . \quad [20];$$

which, when m is an integer, and in the particular cases already mentioned when m is a fraction, becomes

$$\tan.mx = \frac{\text{T}'}{\text{T}},$$

$$\text{or } \tan.mx = \frac{\text{T}}{\text{T}'}.$$

PROP. CIII.

(393.) *To develop the cosine or sine of a multiple arc in descending powers of the cosine of the simple arc.*

This problem was investigated by *Euler*, and subsequently by *Lagrange*, and both obtained the same result, although they proceeded on different principles and by different methods. The series which were the results of their investigations, and which have, even to the present time, been received as general and exact, are the following,

$$2\cos.mx = (2y)^m - m(2y)^{m-2} + \frac{m(m-3)}{1.2}(2y)^{m-4}$$

$$- \frac{m(m-4)(m-5)}{1.2.3}(2y)^{m-6} + \frac{m(m-5)(m-6)(m-7)}{1.2.3.4}(2y)^{m-8}$$

$$\ldots\ldots [21],$$

$$- \ldots\ldots + \ldots\ldots$$

$$+(2y)^{-m}+m(2y)^{-m-2}+\frac{m.\overline{m+3}}{1.2}(2y)^{-m-4},$$

$$+\frac{m(m+4)(m+5)}{1.2.3}(2y)^{-m-6}+\frac{m(m+5)(m\pm 6)(m+7)}{1.2.3.4}(2y)^{-m-8}$$

$$+\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ +\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ .$$

$$\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot$$

where $y = \cos.x$. The series for $\sin.mx$ was deduced from this by differentiation.

In the memoir already cited, *Poinsot* has examined the analysis by which these results were obtained, and shown that it is fallacious, and that the results themselves are false. To render this refutation intelligible, it would be necessary to detail the process by which *Euler* and *Lagrange* established the formulæ, which would lead to investigations unsuited to the purposes of the present treatise. As, however, the results of *Lagrange* have been hitherto universally received as correct, it is proper to make the student aware of the fact of their having been proved erroneous; and if he be desirous to examine the details of the process, he is referred to the memoir itself.

We shall confine ourselves here to that part of the memoir in which the true development of $\cos.mx$ and $\sin.mx$ is investigated.

Let $p = \cos.x$ and $q = \sin.x$. By (366.), we have

$$\cos.mx = p^m\left(1 - A_2\frac{q^2}{p^2} + A_4\frac{q^4}{p^4} - A_6\frac{q^6}{p^6} + \cdot\cdot\cdot\cdot\right)$$

where 1, A_1, A_2, ... are the coefficients of the binomia series, m being the exponent. We have

$$q^2 = 1 - p^2,\quad q^4 = 1 - 2p^2 + p^4, \ldots .$$

Let these values be substituted for q^2, q^4, &c. and let the results be arranged according to the descending powers of p, and we have

$$\cos.mx = Ap^m - Bp^{m-2} + \frac{1}{2}Cp^{m-4} - \frac{1}{1.2.3}Dp^{m-6} + \&c.$$

where

$$\begin{aligned}
A &= 1 + A_2 + A_4 + A_6 \dots \\
B &= A_2 + 2A_4 + 3A_6 + 4A_8 + 5A_{10} \dots \\
\tfrac{1}{2}C &= A_4 + 3A_6 + 6A_8 + 10A_{10} + \dots \\
\frac{1}{1.2.3}D &= A_6 + 4A_8 + 10A_{10} + 20A_{12} \\
\dots &= \dots \\
\dots &= \dots
\end{aligned}$$

The law by which these coefficients are formed is evident, but it is necessary to obtain finite expressions for them as functions of m. For this purpose, let us suppose that the successive terms of the first coefficient A were multiplied by the successive powers of an arbitrary quantity y, so that it becomes

$$1 + A_2 y + A_4 y^2 + A_6 y^3 + \dots$$

$$\text{or } 1 + \frac{m(m-1)}{1.2} y + \frac{m(m-1)(m-2)(m-3)}{1.2.3.4} y^2 + \dots$$

But this last is equivalent to

$$\frac{(1+\sqrt{y})^m + (1-\sqrt{y})^m}{2} = U;$$

so that U becomes equal to A when $y = 1$. It is not difficult to perceive that the other coefficients are what the successive differential coefficients of U taken with respect to y as a variable become when $y = 1$. We have

$$\begin{aligned}
U &= 1 + A_2 y + A_4 y^2 + A_6 y^3 + \dots \\
\frac{dU}{dy} &= A_2 + 2A_4 y + 3A_6 y^2 + 4A_8 y^3 \dots \\
\frac{1}{2}\frac{d^2U}{dy^2} &= A_4 + 3A_6 y + 6A_8 y^2 + \dots \\
\dots &= \dots \\
\dots &= \dots
\end{aligned}$$

When $y = 1$, the second members of these equations become equal severally to A, B, $\frac{1}{2}$C, $\frac{1}{1.2.3}$.D ..... Let the

values of the function U and its successive differential coefficients when $y = 1$ be called Y, Y', Y'', Y''', &c.; we have hence

$$A = Y = \frac{1}{2}\left\{ 2^m + 0^m \right\},$$

$$B = Y' = \frac{1}{2^2}\left\{ m(2^{m-1} - 0^{m-1}) \right\},$$

$$C = Y'' = \frac{1}{2^3}\left\{ m(m-1)(2^{m-2} + 0^{m-2}) - m(2^{m-1} - 0^{m-1}) \right\},$$

$$D = Y''' = \frac{1}{2^4}\left\{ m(m-1)(m-2)(2^{m-3} - 0^{m-3}) \right.$$

$$\left. - 3m(m-1)(2^{m-2} + 0^{m-2}) + 3m(2^{m-1} - 0^{m-1}) \right\},$$

$$E = Y'''' = \frac{1}{2^5}\left\{ m(m-1)(m-2)(m-3)(2^{m-4} + 0^{m-4}) \right.$$

$$-6m(m-1)(m-2)(2^{m-3} - 0^{m-3}) + 15m(m-1)(2^{m-2} + 0^{m-2})$$

$$\left. - 15m(2^{m-1} - 0^{m-1}) \right\}, \text{ \&c. \&c.}$$

In these analytical expressions for the coefficients of the sought series, it is necessary to preserve the terms 0^m, 0^{m-1}, 0^{m-2}, &c. because each of these powers of 0 become either unity, 0, or infinite, according as the exponent of the power is $= 0$, positive or negative.

The true development therefore of $\cos.mx$ in descending powers of $\cos.x$ or p, the angle x being supposed less than a right angle, and only considering a single value of $\cos.mx$ relative to the arc x, is

$$\cos.mx = Yp^m - Y'p^{m-2} + \tfrac{1}{2}Y''p^{m-4} - \frac{1}{2.3}Y'''p^{m-6} + \ldots$$

If m be a positive integer, this series will be finite, since all the terms beyond a certain term will $= 0$, and it will thus give the exact value of $\cos.mx$. Thus when $m = 0$, or $m = 1$, we find that the first coefficient only has a finite value, and all the others $= 0$. For $m = 2$ and $m = 3$, the

first two coefficients are finite, and all the rest $= 0$. For $m = 4$, $m = 5$, there are three terms finite, and all the rest equal nothing; and in general, if m be an even integer, the number of finite terms is $\frac{m}{2} + 1$, and if it be odd, $\frac{m+1}{2}$.

But if m be a fraction, the series never terminates, and the coefficients only continue finite as long as the exponent of 0 which occurs in them is not negative. After this happens, all the succeeding coefficients are infinite. Thus, if m be a fraction between 0 and 1, the first coefficient alone is finite, and all the rest infinite. If m be between 1 and 2, the first two coefficients are finite, and all the rest infinite, and so on. If m be a fraction between $n - 1$ and n, the first n terms are finite, and all the rest infinite. The series therefore in these cases is useless and absurd, and the same happens when m is negative. From whence we may conclude, that the development of the cosine of a multiple arc in descending powers of that of the simple arc is never possible, except when the coefficient of the multiple is a positive integer; and in this case, since the number of terms is finite, the series is nothing more than the series already obtained in ascending powers, the order of the terms being reversed. So that in effect, the only case in which the development by descending powers is possible, it is useless.

It is worthy of remark, that in the analytical expression for the coefficients A, B, $\frac{1}{2}$C, &c. if the powers 0^m, 0^{m-1}, 0^{m-2}, &c. be neglected, the coefficients will be exactly those of the series [21], which has been hitherto considered exact. Whence may be seen the reason why this series gives false values for $\cos. mx$, and also why in the particular case in which m is an integer the value resulting from it will be exact if we retain in it only the positive powers of p, for that is in effect rejecting all that part of the true development which becomes $= 0$.

(394.) The series for $\cos.mx$ in descending powers of $\cos.x$ or p, m being supposed to be an integer, is therefore

$$2\cos.mx=(2p)^m-m(2p)^{m-2}+\frac{m(m-3)}{1.2}(2p)^{m-4}$$
$$-\frac{m(m-4)(m-5)}{1.2.3}(2p)^{m-6}+\frac{m(m-5)(m-6)(m-7)}{1.2.3.4}(2p)^{m-8}$$
$$-\cdots\cdot [22].$$

(395.) To define the law of this series, let the rth term be T,

$$\text{T} = \pm \frac{m(m-r)(m-r-1)\cdots(m-2r+4)(m-2r+3)}{1.2.3\cdots\cdots\cdots(r-1)}(2p)^{m-2(r-1)}.$$

To determine the last term z, let the values of n already found be substituted for r in this formula.

If m be even, let $\frac{m}{2}+1$ be substituted for r, and the result is

$$z=\pm\frac{m\left(\frac{m}{2}-1\right)\left(\frac{m}{2}-2\right)\cdots\cdot 3.2.1}{1.2.3\cdots\cdot\left(\frac{m}{2}-1\right)\frac{m}{2}}(2y)^{m-m}.$$

All the factors of the numerator, except the first, destroy all the factors of the denominator, except the last, and therefore

$$z=\pm 2,$$

+ being taken when $\frac{m}{2}+1$ is odd, and − when $\frac{m}{2}+1$ is even.

If m be odd, let $\frac{m+1}{2}$ be substituted for n, and the result is

$$z=\pm\frac{m\left(\frac{m-1}{2}\right)\left(\frac{m-3}{2}\right)\cdots\cdots 3.2.1}{1.2.3\cdots\cdots\left(\frac{m-3}{2}\right)\left(\frac{m-1}{2}\right)}(2y).$$

The factors of the denominator destroying those of the numerator, except the first, we obtain

$$z = \pm 2my,$$

$+$ being taken when $\frac{m+1}{2}$ is odd, and $-$ when it is even.

PROP. CIV.

(396.) *To develop* sin.mx *in descending powers of* cos.x.

To effect this, it is only necessary to differentiate the series [22]. This being done, and the result divided by $2m$, and observing that $dy = d\cos.x = -\sin.x dx$, we obtain

$$\frac{\sin.mx}{\sin.x} = (2y)^{m-1} - (m-2)(2y)^{m-3} + \frac{(m-3)(m-4)}{1.2}(2y)^{m-5}$$

$$- \dots \quad [23].$$

This development, like the last, is only possible when m is an integer.

When m is an even integer, the number of terms in the series for $2\cos.mx$ being $\frac{m}{2}+1$, and the last term

$$z = \pm 2,$$

it follows, since $dz = 0$, that in the present case, the number of terms must be $\frac{m}{2}$.

The rth term in the present case is evidently

$$\pm \frac{(m-r)(m-r-1)\cdots(m-2r+3)(m-2r+2)}{1.2.3 \cdots\cdots\cdots r-1}(2y)^{m-(2r-1)}.$$

Hence the last term, m being an even integer, may be found by substituting $\frac{m}{2}$ for r in this formula, which gives

$$z = \pm \frac{\frac{m}{2}\left(\frac{m}{2}-1\right)\cdots\cdots 3.2}{2.3\cdots\cdots\left(\frac{m}{2}-1\right)}(2y).$$

$$\therefore z = \pm my.$$

If m be odd, the last term in the series for $2\cos.mx$ being $\pm m(2y)$, that of the series for $\frac{\sin.mx}{\sin.x}$ is

$$z = \pm 1,$$

the number of terms being $\frac{m+1}{2}$, and + being taken when this is odd, and − when it is even.

PROP. CV.

(397.) *To develop the cosine and sine of a multiple arc in descending powers of the sine of the simple arc.*

In [22] and [23] let x be changed into $\frac{\pi}{2} - x$, and the two series being expressed by M and M′, and p being understood to express $\sin.x$ instead of $\cos.x$, we shall have

$$2\cos.m\left(\frac{\pi}{2} - x\right) = \text{M},$$

$$\sin.m\left(\frac{\pi}{2} - x\right) = \cos.x.\text{M}'.$$

In this case, as in the former, m must be an integer.

If m be even,

$$\cos.m\left(\frac{\pi}{2} - x\right) = \pm \cos.mx,$$

$$\sin.m\left(\frac{\pi}{2} - x\right) = \mp \sin.mx,$$

+ being taken when $\frac{1}{2}m$ is even, and − when odd. Hence, in these cases,

$$2\cos.mx = \pm \text{M},$$
$$2\sin.mx = \mp \cos.x.\text{M}'.$$

If m be odd

$$\cos.m\left(\frac{\pi}{2} - x\right) = \pm \sin.mx,$$

$$\sin.m\left(\frac{\pi}{2} - x\right) = \pm \cos.mx,$$

$+$ being used if $\frac{m-1}{2}$ be even, and $-$ if odd.

Hence

$$\sin.mx = \pm \text{M},$$
$$\cos.mx = \pm \cos.x.\text{M}'.$$

SECTION III.

Of the development of a power of the sine or cosine of an arc in a series of sines or cosines of its multiples.

PROP. CVI.

(398.) *To develop* $\cos.^m x$ *in a series of cosines or sines of multiples of* x.

By (358.) we have

$$2\cos.x = e^{x\sqrt{-1}} + e^{-x\sqrt{-1}},$$

$$\therefore 2^m\cos.^m x = (e^{x\sqrt{-1}} + e^{-x\sqrt{-1}})^m.$$

If this be developed by the binomial theorem, we obtain

$$2^m\cos.^m x = e^{mx\sqrt{-1}} + \text{A}e^{(m-2)x\sqrt{-1}} + \text{B}e^{(m-4)x\sqrt{-1}} + \ldots\ldots$$

where

$$1,\ \text{A},\ \text{B},\ \text{C},\ \ldots\ldots$$

are the coefficients of the binomial series.

Eliminating e by the general formula,

$$\cos.mx + \sqrt{-1}\sin.mx = e^{mx\sqrt{-1}},$$

we obtain

$$2^m\cos.^m x = \cos.mx + A\cos.(m-2)x + B\cos.(m-4)x + \cdots\cdot$$
$$+ \sqrt{-1}[\sin.mx + A\sin.(m-2)x + B\sin.(m-4)x + \cdots\cdot].$$

Let the first series be P_x, and the second Q_x, and we have

$$(2\cos.x)^m = P_x + \sqrt{-1}Q_x.$$

Let $\cos.x$ be first supposed to be positive, and in that case $(2\cos.x)^m$ must have at least one real value. Let this be X, and all its other values will be found by multiplying X by the values of $(1)^m$. They are, therefore, all expressed by the formula

$$X(\cos.2mn\pi + \sqrt{-1}\sin.2mn\pi),$$

n being any integer not exceeding the denominator of m.

Also, in

$$(2\cos.x)^m = P_x + \sqrt{-1}Q_x,$$

no change is made in the first member by changing x into $2n\pi + x$, and therefore

$$(2\cos.x)^m = P_{2n\pi+x} + \sqrt{-1}Q_{2n\pi+x}.$$

Hence

$$X\cos.2mn\pi + \sqrt{-1}.X\sin.2mn\pi = P_{2n\pi+x} + \sqrt{-1}Q_{2n\pi+x} \ldots [1].$$

Equating the real and imaginary parts of this equation, we find

$$X = \frac{1}{\cos.2mn\pi}P_{2n\pi+x},\ X = \frac{1}{\sin.2mn\pi}Q_{2n\pi+x} \cdots\cdot [2].$$

Hence it appears that the real and positive value X of $(2\cos.x)^m$ can be indifferently expressed, either in a series of powers of the cosines or sines of the multiples of x, and that the two series differ from one another only in the constant coefficients.

Between the two series thus found, there subsists a constant relation:

$$\frac{\cos.2mn\pi}{\sin.2mn\pi} = \frac{P_{2n\pi+x}}{Q_{2n\pi+x}};$$

By the formulæ [3], [4], it follows that when $\cos.x$ is negative, the real and positive value of $(2\cos.x)^{\frac{m}{n}}$ may be expressed either in a series of sines or cosines of the multiples of x, and that the two developments differ only in the coefficients; and finally, that their ratio is the same for all values of x between $\frac{\pi}{2}$ and $\frac{3\pi}{2}$.

(400.) If m be a positive integer $Q_x = 0$, and we have

$$(2\cos.x)^m = P_x.$$

The number of terms in P_x is $m+1$, being those of the binomial series. Hence the last term must be

$$\cos.(m-2m)x = \cos.mx,$$

which is equal to the first. And, in like manner, the penultimate term is equal to the second, and every pair of terms equidistant from the extremes are equal.

It follows, therefore, that when m is odd, and $\because m+1$ even, the first half of the series S is equal to $\frac{1}{2}(2^m\cos.^m x) = 2^{m-1}\cos.^m x$; and when m is even, and therefore $m+1$ odd, the first $\frac{m}{2}$ terms together with half the $\left(\frac{m}{2}+1\right)$th term is equal to $2^{m-1}\cos.^m x$.

Hence we conclude,

1°. When m is odd,

$$2^{m-1}\cos.^m x = \cos.mx + A\cos.(m-2)x + B\cos.(m-4)x + \cdots\cdot$$

continued to $\frac{m+1}{2}$ terms.

The last term of this series is

$$M\cos.\left[m-2\left(\frac{m+1}{2}-1\right)\right]x = M\cos.x,$$ M being the coefficient of the $\left(\frac{m+1}{2}\right)^{\text{th}}$ term of an expanded binomial. From the law of the binomial series, we have

$$M = \frac{m.m-1.m-2 \ldots\ldots \left(m - \frac{m-3}{2}\right)}{1.2.3 \ldots\ldots\ldots\ldots \frac{m-1}{2}}.$$

This may, however, be reduced to a somewhat simpler form. Let both terms of the fraction be multiplied by $2^{\frac{m-1}{2}}$, the operation being effected on the denominator by doubling each of its factors; the result is

$$M = \frac{m.m-1.m-2 \ldots\ldots \left(m - \frac{m-3}{2}\right)}{2.4.6 \ldots\ldots\ldots\ldots (m-1)}.2^{\frac{m-1}{2}}.$$

Again, multiplying both numerator and denominator by the odd integers from 1 to m inclusive, in order to complete the series of factors in the denominator,

$$M = \frac{m.m-1.m-2 \ldots \left(m - \frac{m-3}{2}\right)}{1.2.3 \ldots\ldots\ldots (m-1)m} 2^{\frac{m-1}{2}}(1.3.5 \ldots . m).$$

Expunging from both numerator and denominator the descending factors from m to $m - \frac{m-3}{2}$ inclusive, we obtain

$$M = \frac{1.3.5 \ldots . m}{1.2.3 \ldots . \frac{m+1}{2}} 2^{\frac{m-1}{2}}.$$

Hence the last term z is

$$z = \frac{1.3.5 \ldots . m}{1.2.3 \ldots . \frac{m+1}{2}} 2^{\frac{m-1}{2}} \cos.x.$$

2°. If m be even,

$$2^{m-1}\cos.^m x = \cos.mx + A\cos.(m-2)x + \ldots .$$

continued to $\frac{m}{2} + 1$ terms, the coefficient of the last term being half that of the $\left(\frac{m}{2} + 1\right)^{\text{th}}$ term of the expanded bi

nomial. Let z be the last term,

$$z = \tfrac{1}{2}\textsc{m}\cos.(m - m)x = \tfrac{1}{2}\textsc{m},$$

$$\textsc{m} = \frac{m.m-1.m-2\ \ldots\ (m-\frac{m}{2}+1)}{1.2.3\ \ldots\ldots\ldots\ \frac{m}{2}}.$$

Multiplying both numerator and denominator by $2^{\frac{m}{2}}$ in the same manner as in the last case, and introducing the deficient factors $1.3.5 \ldots m-1$, we obtain

$$\textsc{m} = \frac{m.m-1.m-2\ \ldots\ (m-\frac{m}{2}+1)}{1.2.3\ \ldots\ldots\ldots\ m}\,2^{\frac{m}{2}}\,(1.3.5 \ldots m-1).$$

Expunging from the numerator and denominator the descending factors from m to $(m - \frac{m}{2} + 1)$ inclusive, we obtain

$$\textsc{m} = \frac{1.3.5\ \ldots\ (m-1)}{1.2.3\ \ldots\ldots\ \frac{m}{2}}\,2^{\frac{m}{2}},$$

$$\therefore z = \frac{1.3.5\ \ldots\ (m-1)}{1.2.3\ \ldots\ldots\ \frac{m}{2}}\,2^{\frac{m}{2}-1},$$

which is the value of the last term.

(401.) The development which has been thus obtained gives the value of the mth power of the cosine of an arc in a series of cosines or sines of its multiples. Similar series for the mth power of the sine may be obtained in a similar way.

By expanding

$$(2\sin.x)^m(\sqrt{-1})^m = \left(e^{x\sqrt{-1}} - e^{-x\sqrt{-1}}\right)^m,$$

and eliminating e by the formula,

$$\cos.mx \pm \sqrt{-1}\,\sin.mx = e^{\pm mx\sqrt{-1}},$$

we obtain

$$(2\sin.x)^m(\sqrt{-1})^m = \cos.mx - \text{A}\cos.(m-2)x + \text{B}\cos.(m-4)x - ..$$
$$+ \sqrt{-1}[\sin.mx - \text{A}\sin.(m-2)x + \text{B}\sin.(m-4)x - ...].$$

If the series be called P_x and Q_x, we have

$$(2\sin.x)^m(-1)^{\frac{m}{2}} = \text{P}_x + \sqrt{-1}\text{Q}_x.$$

This formula being treated in a manner similar to that for $(2\cos.x)^m$, will give similar results.

(402.) If m be a positive integer, the number of terms in each of the series P_x and Q_x will be $m+1$, and one or other of them will $= 0$. We shall consider successively the cases in which m is even and odd.

1°. Let m be even.

The number of terms in Q_x being $(m+1)$ and $\because$ odd, the sign of the last term is by the law of the series $+$, and it is therefore

$$+ \sin.(m - 2m)x = - \sin.mx.$$

The penultimate term is

$$- \text{A}\sin.[m - 2(m-1)]x = - \text{A}\sin.(-m+2)x$$
$$= + \text{A}\sin.(m-2)x,$$

and by continuing the process, it appears that the extreme terms, and those equally distant from them, destroy each other. Hence $\text{Q}_x = 0$, and therefore

$$2^m(\sqrt{-1})^m\sin.^m x = \text{P}_x.$$

But since m is even,

$$(-1)^{\frac{m}{2}} = \pm 1,$$

$+$ being taken when $\frac{m}{2}$ is even, and $-$ when odd. Therefore

$$\pm 2^m\sin.^m x = \text{P}_x.$$

In the same manner as in the former case, it follows that in the series P_x the extreme terms, and those which are equidistant from them, are equal, and have the same sign, and hence, as before, we find

$$\pm 2^{m-1}\sin.^m x = \text{P}_x,$$

the number of terms being $\frac{m}{2} + 1$, and the last term being the same as for $2^{m-1}\cos.^m x$ when m is even.

2°. Let m be an odd integer.

In this case the number of terms being $m + 1$, the sign of the last term of P_x is by the law of the series $-$, and it is therefore

$$-\cos.(m - 2m)x = -\cos.mx,$$

and the penultimate term is

$$+ \text{A}\cos.[m - 2(m - 1)]x = + \text{A}\cos.(m - 2)x,$$

and by continuing the process, it appears that the extreme terms, and those which are equidistant from them, are equal with different signs, and therefore destroy each other. Hence $\text{P}_x = 0$, and

$$2^m(\sqrt{-1})^m\sin.^m x = \sqrt{-1}\text{Q}_x$$

$$\therefore 2^m(\sqrt{-1})^{m-1}\sin.^m x = \text{Q}_x.$$

But since $m - 1$ is even,

$$(\sqrt{-1})^{m-1} = \pm 1,$$

$$\therefore \pm \text{Q}^m\sin.^m x = \text{Q}_x,$$

$+$ being taken when $\frac{m-1}{2}$ is even, and $-$ when it is odd.

In the same manner as before, it may be shown that the extreme terms of Q_x are equal and have the same sign. Hence we find

$$2^{m-1}\sin.^m x = \text{Q}_x$$

continued to $\frac{m+1}{2}$ terms, the last term being

$$z = \frac{1.3.5 \ldots . m}{1.2.3 \ldots . \frac{m+1}{2}} 2^{\frac{m-1}{2}} \sin.x.$$

SECTION IV.

On the summation of certain series of sines, cosines, &c. of multiple arcs.

(403.) When the law of a series is known, the most direct method of obtaining its sum between proposed limits, or, in other words, the sum of any given number of terms commencing at a given term, is by the principles of the Calculus of Differences. This method will be found fully explained in my treatise on the Differential and Integral Calculus, Part IV., Section V. Several examples are given there, and they occur in still greater numbers in the examples on the Calculus of Differences by Herschel, which accompany the examples on the Differential and Integral Calculus by Peacock. As, however, this general method is not considered sufficiently elementary for students at a certain stage of their progress, other contrivances for obtaining the sums of particular series which rest upon more simple principles, and are derivable from the established formulæ of trigonometry, are resorted to.

The defect of this way of obtaining the sums of trigonometrical series is, that it has few or no general principles; that the means of obtaining the sum of one series do not suggest the means of summing another; that there is a different artifice or contrivance, in fact, a different method in each particular case to which we have nothing to direct us but the peculiar nature of the series under examination. These methods, if they can be called so, stand in exactly the same relation to that which is founded upon the principles of the Calculus of Differences, as the various contrivances

of the ancient geometers to draw tangents to curves, to rectify them, &c. do to the general methods of solving these problems by the modern calculus *.

We shall, in the present section, limit our investigations to the summation of a few of the most common series; it being much better for the student who desires an extensive acquaintance with the subject to acquire a knowledge of the necessary parts of the calculus of differences, than to burthen his memory with the trigonometrical artifices necessary to solve such problems otherwise.

PROP. CVII.

(404.) *To find the sum of the sines of a series of arcs which are in arithmetical progression.*

Let the proposed series be

$$\sin.\text{A}+\sin.(\text{A}+x)+\sin.(\text{A}+2x)\cdots\sin.[\text{A}+(n-1)x]=\text{S}\cdots[1].$$

Let both sides be multiplied by $-2\sin.\frac{1}{2}x$. Hence

$$-2\sin.\tfrac{1}{2}x\sin.\text{A}-2\sin.\tfrac{1}{2}x\sin.(\text{A}+x)-2\sin.\tfrac{1}{2}x\sin.(\text{A}+2x)-\cdots-2\sin.\tfrac{1}{2}x\sin.[\text{A}+(n-1)x]=-2\sin.\tfrac{1}{2}x\text{S}.$$

Every term of this series is of the form,

$$-2\sin.\tfrac{1}{2}x\sin.(\text{A}+mx);$$

and by (45.), [9], we have

$$-2\sin.\tfrac{1}{2}x\sin.(\text{A}+mx)=\cos.(\text{A}+\frac{2m+1}{2}x).$$

$$-\cos.(\text{A}+\frac{2m-1}{2}x).$$

By substituting successively 0, 1, 2, 3, $\cdots$ $(n-1)$ for m in this formula, we shall obtain the values of the successive terms of the above series, and it is plain that, except the

---

* See Geometry. Introduction, p. xxiii.

first and last, they will mutually destroy each other, so that the result will be

$$-\cos.(A-\tfrac{1}{2}x)+\cos.(A+\frac{2n-1}{2}x)=-2s\sin.\tfrac{1}{2}x,$$

which by (45.), [9], becomes

$$\sin.(A+\frac{n-1}{2}x)\sin.\frac{n}{2}x=2s\sin.\tfrac{1}{2}x,$$

$$\therefore s=\frac{\sin.(A+\frac{n-1}{2}x)\sin.\frac{n}{2}x}{\sin.\frac{1}{2}x}.$$

(405.) If $A = x$, the series becomes

$$\sin.x+\sin.2x+\sin.3x\ldots.+\sin.nx=s\quad\ldots.\quad[2],$$

and we have

$$s=\frac{\sin.\frac{n+1}{2}x\sin.\frac{n}{2}x}{\sin.\frac{1}{2}x}.$$

(406.) It will be perceived that the contrivance by which the summation of the proposed series has been effected is the conversion of it into another series, every term of which being the double product of two sines, admits of being resolved into two simple cosines with different signs. In this case one of the cosines into which each term is resolved destroys one of the cosines into which the next term is resolved; so that, however numerous the terms of the series may be, the total result can only contain one of the cosines of the first pair, and one of the last pair. If the last cosine continually diminished or approached any value as a limit, as the number of terms in the series increased, we should be entitled to conclude, that the sum of the proposed series continued *ad infinitum* would be expressed by the first cosine and the limiting value of the last. If we assume that

the sum of the series *ad inf.* is expressed by the first term alone, it is equivalent to assuming that the last diminishes without limit.

This, however, is not the case. The last cosine alternately increases and decreases as the arc changes its relation to an exact multiple of the circumference, and therefore the series increases and decreases alternately, and approaches no limiting state, as its terms increase in number *ad inf.* In other words, the series not being convergent, does not admit of having its sum assigned when the number of its terms is infinite.

(407.) If x be commensurable with the circumference or 2π, the series will be *periodic*, that is, after a certain number of terms, the same terms will continually recur. Let the least integers in the ratio of x to 2π be m', n', so that

$$n'x = 2m'\pi.$$

In that case, when the series [1] has been continued to n' terms, the $(n' + 1)$th term will be

$$\sin.(\text{A} + n'x) = \sin.(\text{A} + 2m'\pi) = \sin.\text{A}.$$

In like manner the following term will be

$$\sin.[\text{A} + (n' + 1)x] = \sin.(\text{A} + x + 2m'\pi) = \sin.(\text{A} + x),$$

which is equal to the second term of the series, and so the terms from the $(n' + 1)$th to the $2n'$th inclusive, will be respectively equal to those from the first to the n'th inclusive.

In this case the value of the *period* of the series may be found by substituting $\frac{2m'\pi}{n'}$ for x and n' for n in the value of s already found, which gives

$$s = \frac{\sin.\left(A + \frac{(n'-1)m'\pi}{n'}\right)\sin.m'\pi}{\sin.\frac{m'\pi}{n'}}.$$

If $\frac{m'}{n'}$ be not an integer, the value of s must $= 0$, for $\sin.m'\pi = 0$, and in this case $\sin.\frac{m'\pi}{n'}$ cannot $= 0$. Therefore, in this case, the terms of the period mutually destroy one another; or the whole period might be divided into two periods, the terms of which differ only in their signs. But if $\frac{m'}{n'}$ be an integer, then $\sin.\frac{m'\pi}{n'} = 0$, and the value of s assumes the form $\frac{0}{0}$. In this case x is an exact multiple of 2π, and the terms of the series are all equal to the first term sin.A, which is in this case the period.

(408.) One meaning of the sum of a periodic series continued *ad inf.* would evidently be the product of the value of the period and an infinite integer, or the period added to itself *ad inf.* It may, however, be considered, that if continued *ad inf.*, every term of the period has as strong a claim as the last term to be considered the last term of the series.

The sum of a periodic series, whose period = 0, continued *ad inf.* is therefore susceptible of as many *different* values as there are different terms in its period. If the last term of the series be the first term of a period, that term will be equal to the sum. If the last term be the second term of a period, the sum will be equal to the first two

terms; and if the last term be the third term of a period, the sum will be the first three terms, and so on. It is therefore evident that the variety of values of which the sum is susceptible is limited by the number of terms in the period. Now the sum of the series continued *ad inf.* has been said to be equal to a mean of all its different values, or to the sum of all the different values divided by their number. We shall give an instance of this in the series of which we have already obtained the sum.

If n' be the number of terms in the period, we shall obtain the n' different values of the series s by successively substituting $0, 1, 2, 3, \ldots n'$ for n in the equation

$$\cos.(A-\tfrac{1}{2}x)-\cos.(A+\frac{2n+1}{2}x)=2\sin.\tfrac{1}{2}xs.$$

Let s' be the sum of all the corresponding values of s. Since the sum of all the corresponding values of

$\cos.(A+\frac{2n+1}{2}x)$ is the period which, by hyp. $=0$, we have

$$n'\cos.(A-\tfrac{1}{2}x)=2s'\sin.\tfrac{1}{2}x,$$

$$\therefore \frac{s'}{n'}=\frac{\cos.(A-\frac{1}{2}x)}{2\sin.\frac{1}{2}x}.$$

This is the value for s, which would be obtained by neglecting the last cosine in the investigation in (404.), and it hence appears that we cannot infer that the sum of the series *ad inf.* has this value, except when the series is periodic, and its period $=0$; and then the sum *ad inf.* has this value only in the sense above explained, which is, that the sum *ad inf.* is susceptible of as many different values as there are terms in the period, and that which is found above is its

mean value, or the sum of all its different values divided by their number.

We have enlarged somewhat upon the subject, to remove the misconceptions into which we are aware many students have been led by investigating the sums of various trigonometrical series continued *ad inf.* by the method which we have here explained, and which is never applicable, except under the particular circumstances, and in the particular sense which have been above stated.

PROP. CVIII.

(409.) *To find the sum of the cosines of a series of arcs in arithmetical progression.*

In [1], (404.), let x be changed into $-x$, and A into $\frac{\pi}{2} - \text{A}$, and the series becomes

$$\cos.\text{A} + \cos.(\text{A} + x) + \cos.(\text{A} + 2x) \dots$$
$$+ \cos.[\text{A} + (n-1)x] = \text{S} \dots [3].$$

The same changes being made in the value obtained for the sum, we have

$$\text{S} = \frac{\cos.(\text{A} + \frac{n-1}{2}x)\sin.\frac{n}{2}x}{\sin.\frac{1}{2}x}.$$

If $A = x$, the series becomes

$$\cos.x + \cos.2x + \cos.3x + \ldots\cos.nx = \frac{\cos.\frac{n+1}{2}x\sin.\frac{n}{2}x}{\sin.\frac{1}{2}x} \ldots[4].$$

The observations on the series for the sines continued *ad inf.* also apply here. When x is commensurable with 2π, but not a multiple of it, the mean value of the sum of the series continued *ad inf.* is

$$S = \frac{\cos.\left(\frac{\pi}{2} - A + \frac{1}{2}x\right)}{-2\sin.\frac{1}{2}x} = -\frac{\sin.(A - \frac{1}{2}x)}{2\sin.\frac{1}{2}x},$$

and when $A = x$, under the same restrictions,

$$S = -\tfrac{1}{2}.$$

PROP. CIX.

(410.) *To find the sum of the sines of a series of arcs in arithmetical progression, the terms of the series being taken alternately positive and negative.*

Let the series be

$$\sin.A - \sin.(A + x) + \sin.(A + 2x) - \ldots\ldots \pm\sin.[A + (n-1)x] = S \quad [5].$$

In [1], let x be changed into $\pi + x$, and it becomes identical with this series. The same change being made in the value of the sum, gives

$$S = \frac{\sin.\left(\frac{n-1}{2}\pi + A + \frac{n-1}{2}x\right)\sin.\left(\frac{n}{2}\pi + \frac{n}{2}x\right)}{\cos.\frac{1}{2}x}.$$

If n be even, this becomes

$$s = -\frac{\cos.(A + \frac{n-1}{2}x)\sin.\frac{n}{2}x}{\cos.\frac{1}{2}x},$$

and if n be odd,

$$s = \frac{\sin.(A + \frac{n-1}{2}x)\cos.\frac{n}{2}x}{\cos.\frac{1}{2}x}.$$

If $A = x$,

$$s = -\frac{\cos.\frac{n+1}{2}x\sin.\frac{n}{2}x}{\cos.\frac{1}{2}x} \quad (n \text{ even}),$$

$$s = \frac{\sin.\frac{n+1}{2}x\cos.\frac{n}{2}x}{\cos.\frac{1}{2}x} \quad (n \text{ odd}).$$

PROP. CX.

(411.) *To find the sum of the cosines of a series of arcs in arithmetical progression, the terms being taken alternately positive and negative.*

Let the series be

$$\cos.A - \cos.(A+x) + \cos.(A+2x) - \ldots \pm \cos.[A+(n-1)x] = s \qquad [6].$$

By changing A into $\frac{\pi}{2} + A$ in [5], the series becomes identical with this. Making this change in the values of s, we find

$$s = \frac{\sin.(A + \frac{n-1}{2}x)\sin.\frac{n}{2}x}{\cos.\frac{1}{2}x} \quad (n \text{ even}),$$

$$s = \frac{\cos.(A + \frac{n-1}{2}x)\cos.\frac{n}{2}x}{\cos.\frac{1}{2}x} \quad (n \text{ odd}).$$

If $\text{A} = x$,

$$s = \frac{\sin.\frac{n+1}{2}x\sin.\frac{n}{2}x}{\cos.\frac{1}{2}x} \quad (n \text{ even}),$$

$$s = \frac{\cos.\frac{n+1}{2}x\cos.\frac{n}{2}x}{\cos.\frac{1}{2}x} \quad (n \text{ odd}).$$

(412.) To sum the series

$$\sin.x + 2\sin.2x + 3\sin.3x + \cdots n\sin.nx = s \;\ldots\ldots\; [7].$$

By [4], (409.), we have

$$\cos.x + \cos.2x + \cdots \cos.nx = \frac{\cos.\frac{n+1}{2}x\sin.\frac{n}{2}x}{\sin.\frac{1}{2}x},$$

$$= \frac{\sin.(n+\frac{1}{2})x - \sin.\frac{1}{2}x}{2\sin.\frac{1}{2}x}.$$

Differentiating both sides, we find

$$-\sin.x - 2\sin.2x - 3\sin.3x - \cdots - n\sin.nx$$

$$= \frac{(n+\frac{1}{2})\sin.\frac{1}{2}x\cos.(n+\frac{1}{2})x - \frac{1}{2}\sin.(n+\frac{1}{2})x\cos.\frac{1}{2}x}{2\sin.^2\frac{1}{2}x}.$$

Changing the signs, and reducing the terms by the established formulæ,

$$s = \frac{\frac{1}{2}\sin.nx - n\sin.\frac{1}{2}x\cos.(n+\frac{1}{2})x}{2\sin.^2\frac{1}{2}x},$$

$$s = \frac{(n+1)\sin.nx - n\sin.(n+1)x}{4\sin.^2\frac{1}{2}x}.$$

(413.) To sum the series

$$\cos.x + 2\cos.2x + 3\cos.3x + \cdots n\cos.nx = s \;\ldots\ldots\; [8].$$

By [2], (405.), we have

$$\sin.x + \sin.2x + \cdots \sin.nx = \frac{\sin.\frac{n+1}{2}x\sin.\frac{n}{2}x}{\sin.\frac{1}{2}x}$$

$$= \frac{\cos.\frac{1}{2}x - \cos.(n+\frac{1}{2})x}{2\sin.\frac{1}{2}x}.$$

Differentiating and reducing, we obtain

$$s = \frac{\sin.\tfrac{1}{2}x\left[(n+\tfrac{1}{2})\sin.(n+\tfrac{1}{2})x - \tfrac{1}{2}\sin.\tfrac{1}{2}x\right] - \tfrac{1}{2}\cos.\tfrac{1}{2}x\left[\cos.\tfrac{1}{2}x - \cos.(n+\tfrac{1}{2})x\right]}{2\sin.^2\tfrac{1}{2}x}$$

$$\therefore s = \frac{\cos.nx + 2n\sin.\tfrac{1}{2}x\sin.(n+\tfrac{1}{2})x - 1}{4\sin.^2\tfrac{1}{2}x},$$

$$s = \frac{(n+1)\cos.nx - n\cos.(n+1)x - 1}{4\sin.^2\tfrac{1}{2}x}.$$

(414.) To sum the series

$$\frac{1}{2}\tan.\frac{1}{2}x + \frac{1}{2^2}\tan.\frac{1}{2^2}x + \frac{1}{2^3}\tan.\frac{1}{2^3}x \cdots + \frac{1}{2^n}\tan.\frac{1}{2^n}x = s \cdots\cdots [9].$$

By (56.),

$$\cot.x = \tfrac{1}{2}(\cot.\tfrac{1}{2}x - \tan.\tfrac{1}{2}x),$$

$$\therefore \tfrac{1}{2}\tan.\tfrac{1}{2}x = \tfrac{1}{2}\cot.\tfrac{1}{2}x - \cot.x.$$

Substituting x, $\frac{1}{2}x$, $\frac{1}{4}x$, &c. successively in this equation for x, we obtain

$$\tfrac{1}{2}\tan.\tfrac{1}{2}x = \tfrac{1}{2}\cot.\tfrac{1}{2}x - \cot.x,$$

$$\frac{1}{2^2}\tan.\frac{1}{2^2}x = \frac{1}{2^2}\cot.\frac{1}{2^2}x - \tfrac{1}{2}\cot.\tfrac{1}{2}x,$$

$$\frac{1}{2^3}\tan.\frac{1}{2^3}x = \frac{1}{2^3}\cot.\frac{1}{2^3}x - \frac{1}{2^2}\cot.\frac{1}{2^2}x,$$

. . . .

. . . .

$$\frac{1}{2^n}\tan.\frac{1}{2^n}x = \frac{1}{2^n}\cot.\frac{1}{2^n}x - \frac{1}{2^{n-1}}\cot.\frac{1}{2^{n-1}}x.$$

By adding these, we obtain

$$s = \frac{1}{2^n}\cot.\frac{1}{2^n}x - \cot.x.$$

We shall find the sum of the series continued *ad inf.* by determining the value of $\frac{1}{2^n}\cot.\frac{1}{2^n}x$ when n is infinite. We have

$$\frac{1}{2^n}\cot.\frac{1}{2^n}x = \frac{\cos.\frac{1}{2^n}x}{2^n\sin.\frac{1}{2}x}.$$

The limiting value of $\cos.\frac{1}{2^n}x$ is unity, and that of $\sin.\frac{1}{2^n}x$ is $\frac{x}{2^n}$. Hence the limiting value sought is $\frac{1}{x}$, and therefore when n is infinite,

$$s = \frac{1}{x} - \cot. x.$$

In this case, if we had adopted the method sometimes used of neglecting the last cotangent without any regard to what its value might be, we should have obtained $s = -\cot.x$.

(415.) To find the sum of the series

$\operatorname{cosec}.x + \operatorname{cosec}.2x + \operatorname{cosec}.4x \cdots + \cos.2(n-1)x = s \cdots\cdots$ [10].

By (60.), [49], we have

$$\operatorname{cosec}.x = \cot.\tfrac{1}{2}x - \cot.x;$$

and in general,

$$\operatorname{cosec}.mx = \cot.\tfrac{1}{2}mx - \cot.mx.$$

Substituting $1, 2, 4, \cdots\cdots 2(n-1)$ successively for m, and adding the results, we have

$$s = \cot.\tfrac{1}{2}x - \cot.2(n-1)x.$$

(416.) The Differential and Integral Calculus furnishes means of obtaining the sums of numerous series. In the well known series *

$$lu = \frac{u^1 - u^{-1}}{1} - \frac{u^2 - u^{-2}}{2} + \frac{u^3 - u^{-3}}{3} - \frac{u^4 - u^{-4}}{4} + \cdots\cdots$$

Substitute $e^{x\sqrt{-1}}$ for u, and observe the condition,

$$2\sqrt{-1}\,\sin.mx = e^{mx\sqrt{-1}} - e^{-mx\sqrt{-1}},$$

and we obtain

$\tfrac{1}{2}x = \sin.x - \tfrac{1}{2}\sin.2x + \tfrac{1}{3}\sin.3x - \tfrac{1}{4}\sin.4x + \cdots\cdots$ [11].

In which the arc is expressed in a series of sines of its multiples.

* Differential Calculus (72.)

(417.) By multiplying [11] by dx, and integrating, we obtain

$$\frac{x^2}{2^2} = -\cos.x + \frac{1}{2^2}\cos.2x - \frac{1}{3^2}\cos.3x + \cdots + \text{const.}$$

To determine the constant let $x = 0$. This gives

$$0 = -1 + \frac{1}{2^2} - \frac{1}{3^2} + \cdots + \text{const.}$$

which being subtracted from the former, gives

$$\frac{x^2}{2^2} = \frac{1-\cos.x}{1} - \frac{1-\cos.2x}{2^2} + \frac{1-\cos.3x}{3^2} - \cdots [12].$$

This might be further modified by the formula,

$$1 - \cos.mx = 2\sin.^2\tfrac{1}{2}mx;$$

but we shall not pursue the investigation.

(418.) By multiplying both sides of [12] by dx, and integrating, we obtain

$$\frac{x^3}{3.2^2} = \frac{x - \sin.x}{1} - \frac{2x - \sin.2x}{2^3} + \frac{3x - \sin.3x}{3^3} - \cdots$$

no constant being added, since both sides $= 0$ when $x = 0$.

(419.) The subject of this section might be extended much further without much difficulty to the author, or much benefit to the student.

The true key to these problems is the Calculus of Differences, and those who desire to enter further into the subject will find that the most expeditious and satisfactory way of proceeding is at once to acquire a competent knowledge of that part of analytical science.

SECTION V.

On the resolution of trigonometrical quantities into factors.

(420.) An example of the application of the formulæ of trigonometry to the decomposition of functions into their

factors has been already shown in the investigation of the theorem of *Moivre* and *Cotes* in the first section of this part. We now propose to give some further examples of this principle.

(421.) By (55.) we have

$$\sin.x = 2\sin.\tfrac{1}{2}x\cos.\tfrac{1}{2}x,$$

and in general,

$$\sin.mx = 2\sin.\tfrac{1}{2}mx\cos.\tfrac{1}{2}mx,$$

$$\therefore \frac{\sin.mx}{\sin.\frac{1}{2}mx} = 2\cos.\tfrac{1}{2}mx.$$

By successively substituting $1, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \ldots \frac{1}{2^{n-1}}$ for m in this, and multiplying all the results together, each successive numerator of the first member destroys the preceding denominator, and we obtain

$$\frac{\sin.x}{\sin.\frac{1}{2^n}x} = 2^n\cos.\tfrac{1}{2}x\cos.\frac{1}{2^2}x\cos.\frac{1}{2^3}x \ldots\ldots \cos.\frac{1}{2^n}x \cdot \cdot \cdot \cdot [1].$$

If the resolution into factors be continued *ad inf.*, we shall obtain the result by determining what

$$2^n\sin.\frac{1}{2^n}x$$

becomes when n is infinite. Since the sine of an evanescent arc equals the arc itself, we have, when n is infinite,

$$2^n\sin.\frac{1}{2^n}x = 2^n\frac{1}{2^n}x = x;$$

also the final factor of the second member approaches unity as its limit. Thus we have

$$\frac{\sin.x}{x} = \cos.\tfrac{1}{2}x\cos.\frac{1}{2^2}x\cos.\frac{1}{2^3}x \cdot \cdot \cdot \cdot \textit{ad inf.} \cdot \cdot \cdot \cdot [2],$$

and hence

$$x = \sin.x\sec.\tfrac{1}{2}x\sec.\frac{1}{2^2}x\sec.\frac{1}{2^3}x \cdot \cdot \cdot \cdot \textit{ad inf.}$$

where the arc is expressed as the continued product of the secants of its submultiples.

(422.) The roots of the equation $\sin.x = 0$ are

$$x = 0, \quad x = \pi, \quad x = 2\pi, \text{ \&c.}$$

and in general $x = n\pi$. Since

$$\sin.x = \frac{x}{1} - \frac{x^3}{(3)} + \frac{x^5}{(5)} - \dots$$

we have by the general theory of equations

$$\sin.x = \text{A}x\left(1 - \frac{x}{\pi}\right)\left(1 + \frac{x}{\pi}\right)\left(1 - \frac{x}{2\pi}\right)\left(1 + \frac{x}{2\pi}\right)\left(1 - \frac{x}{3\pi}\right)\left(1 + \frac{x}{3\pi}\right) \dots \textit{ad inf.}$$

when $x = 0$,

$$\frac{\sin.x}{x} = 1 = \text{A}.$$

Hence

$$\sin.x = x\left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{2^2\pi^2}\right)\left(1 - \frac{x^2}{3^2\pi^2}\right) \dots \textit{ad inf.} \quad [3].$$

In a similar manner from the series

$$\cos.x = 1 - \frac{x^2}{(2)} + \frac{x^4}{(4)} - \dots$$

we deduce *

$$\cos.x = \left(1 - \frac{4x^2}{\pi^2}\right)\left(1 - \frac{4x^2}{3^2\pi^2}\right)\left(1 - \frac{4x^2}{5^2\pi^2}\right) \dots \textit{ad inf.} \quad [4].$$

(423.) If $x = \frac{m\pi}{n}$, we shall have

$$\sin.\frac{m\pi}{n} = \frac{m\pi}{n}\left(1 - \frac{m^2}{n^2}\right)\left(1 - \frac{m^2}{4n^2}\right)\left(1 - \frac{m^2}{9n^2}\right) \dots \quad [5],$$

$$\cos.\frac{m\pi}{n} = \left(1 - \frac{4m^2}{n^2}\right)\left(1 - \frac{4m^2}{9n^2}\right)\left(1 - \frac{4m^2}{25n^2}\right) \dots \quad [6].$$

---

* The method used here for resolving $\sin.x$ and $\cos.x$ into a continued product is pointed out by *Lacroix*, Cal. Diff. of Int. Tom. III., p. 440. The contents of the present section have been selected chiefly from chap. vi. of that volume.

If in [5] $\frac{m}{n}$ be changed into $\frac{1}{2}-\frac{m}{n}$, they become

$$\cos.\frac{m\pi}{n}=\pi\left(\frac{1}{2}-\frac{m}{n}\right)\frac{\left(\frac{1}{2}+\frac{m}{n}\right)\left(\frac{3}{2}-\frac{m}{n}\right)}{1.1}\frac{\left(\frac{3}{2}+\frac{m}{n}\right)\left(\frac{5}{2}-\frac{m}{n}\right)\dots\dots}{2.2\dots\dots}$$

$$=\frac{\pi\left(1-\frac{2m}{n}\right)}{2}\frac{\left(1+\frac{2m}{n}\right)\left(3-\frac{2m}{n}\right)}{2.2}\frac{\left(3+\frac{2m}{n}\right)\left(5-\frac{2m}{n}\right)}{4.4}\frac{\left(5+\frac{2m}{n}\right)}{6.6\dots\dots}$$

$$=\frac{\pi}{2}\cdot\frac{1.1.3.3.5.5\dots}{2.2.4.4.6.6\dots}\left(1-\frac{4m^2}{n^2}\right)\left(1-\frac{4m^2}{9n^2}\right)\left(1-\frac{4m^2}{25n^2}\right)\dots\dots$$

which by [6] gives

$$\frac{\pi}{2}=\frac{2.2.4.4.6.6.8.8\dots\dots}{1.1.3.3.5.5.7.7\dots\dots}\quad\dots\dots\quad[7].$$

The same conclusion might be deduced in a similar way by substituting $\frac{1}{2}-\frac{m}{n}$ for $\frac{m}{n}$ in [6].

(424.) Since

$$e^{x\sqrt{-1}}-e^{-x\sqrt{-1}}=2\sqrt{-1}\sin.x,$$

$$e^{x\sqrt{-1}}+e^{-x\sqrt{-1}}=2\cos.x;$$

we have by changing $x\sqrt{-1}$ into x and x into $\frac{x}{\sqrt{-1}}$,

$$e^x-e^{-x}=2\sqrt{-1}\sin.\frac{x}{\sqrt{-1}},$$

$$e^x+e^{-x}=2\cos.\frac{x}{\sqrt{-1}}.$$

Hence we find by [3] and [4],

$$\tfrac{1}{2}(e^x-e^{-x})=\frac{x}{(1)}+\frac{x^3}{(3)}+\frac{x^5}{(5)}$$

$$=x\left(1+\frac{x^2}{\pi^2}\right)\left(1+\frac{x^2}{4\pi^2}\right)\left(1+\frac{x^2}{9\pi^2}\right)\dots\dots[8],$$

$$\tfrac{1}{2}(e^x+e^{-x})=1+\frac{x^2}{(2)}+\frac{x^4}{(4)}+\frac{x^6}{(6)}+\ \ldots\ldots$$

$$=\left(1+\frac{4x^2}{\pi^2}\right)\left(1+\frac{4x^2}{9\pi^2}\right)\left(1+\frac{4x^2}{25\pi^2}\right)\ \ldots\ldots\ [9].$$

(425.) By these formulæ, the more general expression

$$\tfrac{1}{2}(e^x \pm e^{\pm y})$$

may be resolved into factors. For we have

1°. $\tfrac{1}{2}(e^x+e^y)=e^{\frac{1}{2}(x+y)}\,.\,\tfrac{1}{2}(e^{\frac{1}{2}(x-y)}+e^{-\frac{1}{2}(x-y)})$

$$=e^{\frac{1}{2}(x+y)}\left(1+\frac{(x-y)^2}{\pi^2}\right)\left(1+\frac{(x-y)^2}{9\pi^2}\right)\left(1+\frac{(x-y)^2}{25\pi^2}\right)\ldots\ldots$$

2°. $\tfrac{1}{2}(e^x+e^{-y})=e^{\frac{1}{2}(x-y)}\,.\,\tfrac{1}{2}(e^{\frac{1}{2}(x+y)}+e^{-\frac{1}{2}(x+y)})$

$$=e^{\frac{1}{2}(x-y)}\left(1+\frac{(x+y)^2}{\pi^2}\right)\left(1+\frac{(x+y)^2}{9\pi^2}\right)\left(1+\frac{(x+y)^2}{25\pi^2}\right)\ldots\ldots$$

3°. $\tfrac{1}{2}(e^x-e^y)=e^{\frac{1}{2}(x+y)}\,.\,\tfrac{1}{2}(e^{\frac{1}{2}(x-y)}-e^{-\frac{1}{2}(x-y)})$

$$=e^{\frac{1}{2}(x+y)}\,\frac{x-y}{2}\left(1+\frac{(x-y)^2}{4\pi^2}\right)\left(1+\frac{(x-y)^2}{16\pi^2}\right)\left(1+\frac{(x-y)^2}{36\pi^2}\right)\ldots\ldots$$

4°. $\tfrac{1}{2}(e^x-e^{-y})=e^{\frac{1}{2}(x-y)}\,.\,\tfrac{1}{2}(e^{\frac{1}{2}(x+y)}-e^{-\frac{1}{2}(x+y)})$

$$=e^{\frac{1}{2}(x-y)}\,\frac{x+y}{2}\left(1+\frac{(x+y)^2}{4\pi^2}\right)\left(1+\frac{(x+y)^2}{16\pi^2}\right)\ \ldots\ldots$$

SECTION VI.

The determination of the roots of certain numerical equations by the aid of trigonometrical tables.

(426.) The formulæ which have been established in the preceding sections, and which express the relations between the sines and cosines of multiple arcs and those of the simple ones, furnish, with the aid of trigonometrical tables, easy methods for the solution of certain numerical equations, the roots of which could only be found algebraically by a tedious and elaborate process.

The principle of these methods consists in finding an equation between trigonometrical functions of an angle of the same form as the numerical equation proposed for solution, and then assigning such a value to the angle as will render the coefficients or known terms of the proposed equation severally equal to the analogous terms of the trigonometrical equation. It may be then inferred that the value of the unknown quantity in the proposed equation will be equal to that of the corresponding quantity in the trigonometrical equation. In this investigation it should not be forgotten that the radius is arbitrary, and that we have the power to assign to it such a value as may be found necessary to identify the two equations. The principle of this method of finding the roots of numerical equations will be easily comprehended by its application to examples.

All equations of the second degree must come under the form

$$x^2 + px = q,$$

where p and q may have any values positive or negative.

If x', x'', be the roots of this equation, we have

$$x' = -\tfrac{1}{2}p + \sqrt{\tfrac{1}{4}p^2+q} = -\tfrac{1}{2}p\left\{1 - \sqrt{1+\frac{4q}{p^2}}\right\},$$

$$x'' = -\tfrac{1}{2}p - \sqrt{\tfrac{1}{4}p^2+q} = -\tfrac{1}{2}p\left\{1 + \sqrt{1+\frac{4q}{p^2}}\right\}.$$

We shall consider separately the cases where q is positive and negative.

1°. If $q > 0$, let $\tan.^2\phi = \dfrac{4q}{p^2}$, $\therefore \dfrac{p}{2} = \sqrt{q}\cot.\phi$.

Hence

$$x' = -\sqrt{q}\cot.\phi(1 - \sec.\phi) = -\sqrt{q}\,\frac{\cos.\phi - 1}{\sin.\phi},$$

$$x'' = -\sqrt{q}\,\frac{\cos.\phi + 1}{\sin.\phi}.$$

Hence we find

$$x' = \sqrt{q}\,\tan.\tfrac{1}{2}\phi,$$
$$x'' = -\sqrt{q}\,\cot.\tfrac{1}{2}\phi.$$

It is always possible to find an angle (ϕ) whose tangent is equal to $\frac{4q}{p^2}$, because the tangent of an angle is susceptible of all values from 0 to infinity.

2°. If $q < 0$, the values x', x'', become

$$x' = -\tfrac{1}{2}p\left\{1 - \sqrt{1 - \frac{4q}{p^2}}\right\},$$
$$x'' = -\tfrac{1}{2}p\left\{1 + \sqrt{1 - \frac{4q}{p^2}}\right\}.$$

Let $\sin.^2\phi = \frac{4q}{p^2}$, $\because$

$$x' = -\tfrac{1}{2}p(1 - \cos.\phi) = -p\sin.^2\tfrac{1}{2}\phi = -\sqrt{q}\,\tan.\tfrac{1}{2}\phi,$$
$$x'' = -\tfrac{1}{2}p(1 + \cos.\phi) = -p\cos.^2\tfrac{1}{2}\phi = -\sqrt{q}\,\cot.\tfrac{1}{2}\phi.$$

In this case, if $\frac{4q}{p^2} < 1$, it is always possible to find an angle whose sine is equal to $\frac{2\sqrt{q}}{p}$, and if $\frac{4q}{p^2} > 1$, the roots are both imaginary.

In each case it is obvious that the sum of the roots is $-p$, and their product q, which are well known properties.

(427.) The most general form for equations of the third order is

$$x^3 + \text{A}x^2 + \text{B}x + \text{C} = 0.$$

But since the second term can always be removed by a transformation determined in the elements of algebra, we shall consider the equation reduced to the form

$$x^3 + px + q = 0.$$

To solve this a transformation is necessary. Let

$$x = z - \frac{p}{3z};$$

which being substituted in the equation, it becomes

$$x^6 + qx^3 = \frac{p^3}{27}.$$

This being a quadratic equation with respect to x^3 may be solved, and gives

$$x^3 = -\tfrac{1}{2}q \pm \sqrt{\frac{p^3}{27} + \tfrac{1}{4}q^2},$$

$$\therefore x = \left(-\tfrac{1}{2}q \pm \sqrt{\frac{p^3}{27} + \tfrac{1}{4}q^2}\right)^{\frac{1}{3}},$$

$$\therefore x = \left(-\tfrac{1}{2}q + \sqrt{\frac{p^3}{27} + \tfrac{1}{4}q^2}\right)^{\frac{1}{3}} + \frac{\tfrac{1}{3}p}{\left(-\tfrac{1}{2}q + \sqrt{\frac{p^3}{27} + \tfrac{1}{4}q^2}\right)^{\frac{1}{3}}}.$$

Multiplying both numerator and denominator by

$$\left(-\tfrac{1}{2}q - \sqrt{\frac{p^3}{27} + \tfrac{1}{4}q}\right)^{\frac{1}{3}},$$

we obtain *

$$x = \left(-\tfrac{1}{2}q + \sqrt{\frac{p^3}{27} + \tfrac{1}{4}q^2}\right)^{\frac{1}{3}} + \left(-\tfrac{1}{2}q - \sqrt{\frac{p^3}{27} + \tfrac{1}{4}q^2}\right)^{\frac{1}{3}}.$$

In this formula p and q may have any values positive or negative. It will be necessary to consider separately the cases where p is positive and negative.

1°. If $p > 0$.

Let $\tan.^2\phi = \frac{4p^3}{27q^2}$, which is always possible since $p > 0$. Hence

* This is *Cardan's rule.* It fails, as will be seen, when $\frac{p^3}{27} > \frac{q^2}{4}$.

$$x = \sqrt[3]{\tfrac{1}{2}q(-1 + \sec\varphi)} + \sqrt[3]{\tfrac{1}{2}q(-1 - \sec.\varphi)}.$$

Let $2\sqrt{\tfrac{1}{3}p} = r$, $\because p = \tfrac{3}{4}r^2$, and $q = \dfrac{r^3}{4\tan.\varphi}$. Hence we obtain

$$x = \tfrac{1}{2}r\left\{\sqrt[3]{\frac{1-\cos.\varphi}{\sin.\varphi}} - \sqrt[3]{\frac{1+\cos.\varphi}{\sin.\varphi}}\right\},$$

$$\because x = \tfrac{1}{2}r(\sqrt[3]{\tan.\tfrac{1}{2}\varphi} - \sqrt[3]{\cot.\tfrac{1}{2}\varphi}).$$

Let $\tan.\theta = \sqrt[3]{\tan.\tfrac{1}{2}\varphi}$, $\because \cot.\theta = \sqrt[3]{\cot.\tfrac{1}{2}\varphi}$. Hence we obtain

$$x = -r \times \tfrac{1}{2}(\cot.\theta - \tan.\theta) = -r\cot.2\theta.$$

But $r = \dfrac{2}{\sqrt{3}}\sqrt{p}$, $\because x = -\dfrac{2\sqrt{p}}{\sqrt{3}}\cot.2\theta$.

The root may therefore be computed by means of the three equations,

$$\tan.\varphi = \frac{2\sqrt[3]{p}}{3^{\frac{1}{3}}.q},$$

$$\tan.\theta = \sqrt[3]{\tan.\tfrac{1}{2}\varphi},$$

$$x = -2\sqrt{\tfrac{1}{3}p}\cot.2\theta.$$

In this case there is but one real root. The imaginary roots may be found by means of the two imaginary cube roots of unity determined in (368.)

If q were negative, the results would be similar, except that the second member of the last equation would be positive, and we should have

$$x = 2\sqrt{\tfrac{1}{3}p}\cot.2\theta.$$

2°. If $p < 0$, the formula becomes

$$x = \left(-\tfrac{1}{2}q + \tfrac{1}{2}q\sqrt{1 - \frac{4p^3}{27q^2}}\right)^{\frac{1}{3}} + \left(-\tfrac{1}{2}q - \tfrac{1}{2}q\sqrt{1 - \frac{4p^3}{27q^2}}\right)^{\frac{1}{3}}.$$

It will be necessary to consider separately the cases where $\frac{4p^3}{27q^2} < 1$ and > 1.

If $\frac{4p^3}{27q^2} < 1$, let

$$\sin.^2\varphi = \frac{4p^3}{27q^2}.$$

Hence

$$x = [\tfrac{1}{2}q(-1 + \cos.\varphi)]^{\frac{1}{3}} + [\tfrac{1}{2}q(-1 - \cos.\varphi)]^{\frac{1}{3}}.$$

Let $r^2 = \frac{4p}{3}$, $\because q = \frac{r^3}{4\sin.\varphi}$. Hence

$$x = -\tfrac{1}{2}r(\sqrt[3]{\tan.\tfrac{1}{2}\varphi} + \sqrt[3]{\cot.\tfrac{1}{2}\varphi})$$

Let $\tan.\theta = \sqrt[3]{\tan.\tfrac{1}{2}\varphi}$, $\because$

$$x = -\tfrac{1}{2}r(\tan.\theta + \cot.\theta) = -r\operatorname{cosec}.2\theta.$$

Hence the value of x is determined by the equations,

$$\sin.\varphi = \frac{2\sqrt{p^3}}{q\sqrt{27}},$$

$$\tan.\theta = \sqrt[3]{\tan.\tfrac{1}{2}\varphi},$$

$$x = -2\sqrt{\tfrac{1}{3}p}\operatorname{cosec}.2\theta.$$

In this case also the other two roots are imaginary.

If q be negative, the second member of the last equation will be positive, and we shall have

$$x = 2\sqrt{\tfrac{1}{3}p}\operatorname{cosec}.2\theta.$$

If $\frac{4p^3}{27q^2} > 1$, there is no angle whose sine is equal to it, and the analytical formula for the roots contains imaginary radicals, although it is known that in this case the three roots are real. This is what is called in algebra the *irreducible case*. To obtain its solution by trigonometry, we need only compare the equation

$$x^3 - px \pm q = 0$$

with a trigonometrical equation of the same form derived from the methods of expressing the sines and cosines of mul-

tiple arcs in terms of the powers of those of the simple arc. By (391.), we have

$$\sin.3\varphi = 3\sin.\varphi - 4\sin.^3\varphi,$$

which, by supplying the radius r, becomes

$$r^2\sin.3\varphi = 3r^2\sin.\varphi - 4\sin.^3\varphi,$$

$$\therefore \sin.^3\varphi - \tfrac{3}{4}r^2\sin.\varphi + \tfrac{1}{4}r^2\sin.3\varphi = 0.$$

By comparing this with the proposed equation, it is plain that if $p = \frac{3}{4}r^2$ and $q = \frac{1}{4}r^2\sin.3\varphi$, we shall have $x = \sin.\varphi$. In this case, therefore, the value of x is determined by the equations

$$\sin.3\varphi = \frac{3q}{p},$$

$$x = \sin.\varphi.$$

In this case the sines are related to the radius $2\sqrt{\frac{1}{3}p}$. Reducing them to the radius unity, we have

$$\sin.3\varphi = \frac{3q}{p} \cdot \frac{1}{2\sqrt{\frac{1}{3}p}}.$$

$$x = \frac{1}{2\sqrt{\frac{1}{3}p}}\sin.\varphi.$$

The value of 3φ being determined by the former equation, the sine of one-third of that value substituted for $\sin.\varphi$ in the latter will determine the value of x. But here it must be remarked, that there are an infinite variety of values of 3φ which will satisfy the former equation. The question really to be solved by the former equation is to find an angle (3φ), of which the sine has a given value less than unity.

Let (3φ) be an acute angle satisfying this condition. It is evident that the same condition will be satisfied by

$$\pi - 3\varphi, \quad 2\pi + 3\varphi,$$

$$3\pi - 3\varphi, \quad 4\pi + 3\varphi,$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

and in general by the angles

$$(2n + 1)\pi - 3\varphi, \quad 2n\pi + 3\varphi.$$

Hence the sines of the third parts of these arcs being

verally substituted in the second equation, give so many values of x. Here then an apparent difficulty arises. If the number of angles which satisfy the first equation be unlimited, will not those which satisfy the second be also unlimited, and so the given equation will have an unlimited number of roots? To remove this difficulty, it should be considered, that although there may be an unlimited number of angles to be substituted in the second equation, yet it by no means follows that the sines of these angles have all different values. The angles to be substituted must be of one or other of the forms

$$\frac{2n+1}{3}\pi - \phi, \quad \frac{2n}{3}\pi + \phi.$$

Since n is an integer, the number $\frac{n}{3}$ must be one or other of the forms

$$m, \quad m + \tfrac{1}{3}, \quad m + \tfrac{2}{3};$$

m being also an integer.

If $\frac{n}{3} = m$, we have

$$\frac{2n+1}{3}\pi - \phi = 2m\pi + (\tfrac{1}{3}\pi - \phi),$$

$$\frac{2n}{3}\pi + \phi = 2m\pi + \phi,$$

$$\therefore \sin.\left(\frac{2n+1}{3}\pi - \phi\right) = \sin.(\tfrac{1}{3}\pi - \phi),$$

$$\sin.\left(\frac{2n}{3}\pi + \phi\right) = \sin.\phi.$$

If $\frac{n}{3} = m + \frac{1}{3}$, we have

$$\frac{2n+1}{3}\pi - \phi = (2m+1)\pi - \phi,$$

$$\frac{2n}{3}\pi + \phi = 2m\pi + (\tfrac{2}{3}\pi + \phi),$$

$$\therefore \sin.\left(\frac{2n+1}{3}\pi - \phi\right) = \sin.\phi,$$

$$\sin.\left(\frac{2n}{3}\pi + \phi\right) = \sin.(\tfrac{2}{3}\pi + \phi) = \sin.(\tfrac{1}{3}\pi - \phi).$$

If $\frac{n}{3} = m + \frac{2}{3}$, we have

$$\frac{2n+1}{3}\pi - \phi = (2m+1)\pi + (\tfrac{2}{3}\pi - \phi),$$

$$\frac{2n}{3}\pi + \phi = (2m+1)\pi + (\tfrac{1}{3}\pi + \phi),$$

$$\therefore \sin.\left(\frac{2n+1}{3}\pi - \phi\right) = -\sin.(\tfrac{2}{3}\pi - \phi) = -\sin.(\tfrac{1}{3}\pi + \phi),$$

$$\sin.\left(\frac{2n}{3}\pi + \phi\right) = -\sin.(\tfrac{1}{3}\pi + \phi).$$

Amongst these values there are but three which differ, and which therefore correspond to the three roots of the equation, which are in this case all real, and represented severally by

$$x = \frac{1}{2\sqrt{\frac{1}{3}p}}\sin.\phi,$$

$$x = \frac{1}{2\sqrt{\frac{1}{3}p}}\sin.(60^\circ - \phi),$$

$$x = -\frac{1}{2\sqrt{\frac{1}{3}p}}\sin.(60^\circ + \phi).$$

Note on Art. 180.

From this proposition *Legendre* deduces the following:

"Let S be the number of solid angles in a polyedron, H the number of its faces, A the number of its edges; then, in all cases, we shall have $S + H = A + 2$.

"Within the polyedron, take a point, from which draw straight lines to the vertices of all its angles; conceive next, that from the same point as a centre, a spherical surface is described, meeting all these straight lines in as many points; join these points by arcs of great circles, so as to form on the surface of the sphere polygons corresponding in position and number with the faces of the polyedron. Let ABCDE be one of these polygons, n the number of its sides; its surface will be $s - 2n + 4$, s being the sum of the angles A, B, C, D, E. If the surface of each polygon is estimated in a similar manner, and afterwards the whole are added together, we shall find their sum, or the surface of the sphere represented by 8, to be equal to the sum of all the angles in the polygons, *minus* twice the number of their sides, *plus* 4 taken as many times as there are faces. Now, since all the angles which lie round any one point A are equal to four right angles, the sum of all the angles in the polygons must be equal to 4 taken as many times as there are solid angles; it is therefore equal to 4S. Also, twice the number of sides AB, BC, CD, &c. is equal to four times the number of edges, or to 4A; because the same edge is always a side in two faces. Hence we have $8 = 4S - 4A + 4H$; or dividing all by 4, we have $2 = S - A + H$; hence $S + H = A + 2$.

"*Cor.* From this it follows, that *the sum of all the plane angles, which form the solid angles of a polyedron, is equal to as many times four right angles as there are units in* $S - 2$, S *being the number of solid angles in the polyedron.*

"For, examining a face the number of whose sides is n, the sum of the angles in this face (25. I.) will be $2n - 4$ right angles. But the sum of these $2n$'s, or twice the number of sides in all the faces, will be 4A; and 4 taken as many times as there are faces will be 4H: hence the sum of the angles in all the faces is $4A - 4H$. Now, by the Theorem just demonstrated, we have $A - H = S - 2$, and consequently $4A - 4H = 4(S - 2)$. Hence *the sum of all the plane angles, &c.**

* "This theorem, which Euler first proved, in the Memoirs of Petersburgh, *anno* 1758, presents several consequences worthy of being developed.

"*First.* Let a be the number of triangles, b the number of quadrilaterals, c the number of pentagons, &c., composing the surface of a polyedron; the total number of faces will be $a + b + c + d +$ &c.; and the total number of their sides will be $3a + 4b + 5c + 6d +$ &c. This latter number is twice that of the edges, since the same edge belongs at once to two faces; hence we shall have

$$\text{H} = a + b + c + d + \&\text{c.}$$
$$2\text{A} = 3a + 4b + 5c + 6d + \&\text{c.}$$

And since, by the theorem in question, $\text{S} + \text{H} = \text{A} + 2$, we obtain

$$2\text{S} = 4 + a + 2b + 3c + 4d + \&\text{c.}$$

The first thing which strikes us in these values is, that the number of faces having an odd number of sides is always even.

"For the sake of brevity, we may put $\omega = b + 2c + 3d +$ &c.; we shall then have

$$\text{A} = \tfrac{3}{2}\text{H} + \tfrac{1}{2}\omega,$$
$$\text{S} = 2 + \tfrac{1}{2}\text{H} + \tfrac{1}{2}\omega.$$

Thus in every polyedron, we have always $\text{A} > \tfrac{3}{2}\text{H}$, and $\text{S} > 2 + \tfrac{1}{2}\text{H}$, it being observed that the sign $>$ does not exclude equality if we should ever have $\omega = 0$.

"The number of all the plane angles in the polyedron is 2A, that of the solid angles is S; so that the mean number of plane angles which go to the formation of a solid angle is $\dfrac{2\text{A}}{\text{S}}$.

"This number cannot be less than 3, because at least three plane angles are required to form a solid angle; hence we must have $2\text{A} > 3\text{S}$, the sign $>$ not excluding equality. If in place of A and S, we substitute their values in terms of H and ω, we shall have $3\text{H} + \omega > 6 + \tfrac{3}{2}\text{H} + \tfrac{3}{2}\omega$, or $3\text{H} > 12 + \omega$. Bring back the values of H and ω into terms of a, b, c, &c., we shall have

$$3a + 2b + c > 12 + e + 2f + 3g + \&\text{c.};$$

from which it appears that a, b, c cannot all be nothing at once, and that consequently there exists no polyedron all whose faces have each more than five sides.

"Since we have $\text{H} > 4 + \tfrac{1}{3}\omega$, by substituting the values of S and A we get $\text{S} > 4 + \tfrac{2}{3}\omega$, and $\text{A} > 6 + \omega$. But at the same time, we have $\omega < 3\text{H} - 12$; and from it there result $\text{S} < 2\text{H} - 4$, and $\text{A} < 3\text{H} - 6$, where it must be recollected that the signs $>$ and $<$ do not exclude equality. These limits are observed in all polyedrons generally.

"*Second.* Suppose $2\text{A} > 4\text{S}$, which is true of a multitude of polyedrons, and particularly of those which have all their solid angles formed by four planes or more; we shall in this case have $\text{H} > 8 + \omega$, or by substituting,

$$a > 8 + c + 2d + 3e + \&\text{c.}$$

Hence the solid must at least have eight triangular faces; the limit $\text{H} > 8 + \omega$ gives $\text{S} > 6 + \omega$, and $\text{A} > 12 + 2\omega$. But we have, at the same time, $\omega > \text{H} - 8$; and from this there result $\text{S} > \text{H} - 2$, and $\text{A} > 2\text{H} - 4$.

"*Third.* Suppose $2\text{A} > 5\text{S}$, which, among other polyedrons, includes all such as have each of their solid angles formed by five planes at least; there will result from it $\text{H} > 20 + 3\omega$, or

$$a > 20 + 2b + 5c + 8d + \&\text{c.}$$

We shall at the same time have $\text{S} > 12 + 2\omega$, and $\text{A} > 30 + 5\omega$; and, lastly, from $\omega > \tfrac{1}{3}(\text{H} - 20)$, we shall deduce the limits $\text{S} > \tfrac{2}{3}(\text{H} - 2)$, and $\text{A} > \tfrac{5}{3}(\text{H} - 2)$.

"We cannot suppose $2\text{A} = 6\text{S}$: because we have $2\text{A} + 2\omega + 12 = 6\text{S}$ generally; hence there is no polyedron which has all its solid angles formed of six planes or more; and accordingly, the least value which each plane angle

one with another, could have, would be the angle of an equilateral triangle; and six of such angles make four right angles, which is too much for a solid angle.

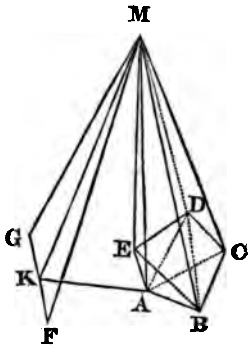
"*Fourth.* Let us examine a polyedron whose faces are all triangular; we shall have $a = 0$, which will give $A = \frac{3}{2}H$, and $S = 2 + \frac{1}{2}H$. Suppose farther, that the solid angles of the polyedron are in part quintuple, in part sextuple; let p be the number of the quintuple solid angles, q of sextuple; we shall have $S = p + q$ and $2A = 5p + 6q$, which give $6S - 2A = p$: but we have besides $A = \frac{3}{2}H$, and $S = 2 + \frac{1}{2}H$; hence $p = 6S - 2A = 12$. Hence *if a polyedron has all its faces triangular, and if its solid angles are in part quintuple, in part sextuple, the number of quintuple solid angles will always amount to* 12. The sextuple may be in any number whatever: thus, leaving q undetermined, we shall have all those solid angles $S = 12 + q$, $H = 20 + 2q$, $A = 30 + 3q$.

"We shall finish these applications by investigating the number of conditions or data necessary for determining any polyedron; an interesting question, which does not yet seem to have been resolved.

"Suppose, first, that the polyedron is *of a determinate kind*, in other words, that we know the number of its faces, the number of their sides individually, and their arrangement with regard to one another. We therefore know the numbers, H, S, A, and likewise a, b, c, d, &c.; we only want farther to discover the actual number of given quantities, lines or angles, by means of which the polyedron may be constructed and determined.

"Let us examine one of the polyedron's faces, which we shall regard as its base. Suppose n to be the number of its sides; there will be $2n - 3$ data required to determine this base. The solid angles out of this base amount in number to $S - n$: the vertex of each solid angle requires three data for determining it; hence the position of $S - n$ vertices will require $3S - 3n$; to which adding the $2n - 3$ data of the base, we shall have in all $3S - n - 3$. But this number in general is too great; it must be diminished by the number of conditions necessary for making the vertices which correspond to the same face lie all in one plane. We have called the number of sides in the base n; let us in like manner call the number of sides in the other faces n', n'', &c. Three points determine a plane; hence whatever more than 3 are found in each of the numbers n', n'', &c., will give just so many conditions for making the different vertices lie in the planes of the faces to which they belong; and the total number of conditions will be equal to the sum $(n' - 3) + (n'' - 3) + (n''' - 3) + \&c$. But the number of terms in this series is $H - 1$; and, moreover, $n + n' + n'' + \&c. = 2A$: hence the sum of the series will be $2A - n - 3(H - 1)$. From this sum take away $3S - n - 3$; there will remain $3S - 2A + 3H - 6$, a quantity, which by reason of $S + H = A + 2$, may be reduced to A. Hence *the number of data necessary for determining a polyedron, among all those of the same species, is equal to the number of its edges.*

"Observe, however, that the data here spoken of must not be taken at random among the lines and angles which constitute the elements of the polyedron; for although there were as many equations as unknown quantities, it might happen that certain relations between the known quantities might render the problem indeterminate. Thus from the theorem just discovered, it might seem that a knowledge of the edges alone would be enough for determining the polyedron; yet there are cases in which this knowledge of itself is not sufficient. If, for example, any prism not triangular were given, an infinite number of other prisms might be formed having edges equal and placed in the same manner. For, whenever the base has more than three sides, the angles may be changed though the same sides are retained, and thus the base may have an infinite number of different forms; also the position of the prism's longitudinal edge with regard to the plane of the base may be changed; finally, these two changes may be combined with each other; and from every new arrangement, a new prism will result still having its edges or sides unchanged. From all which, it is clear, that in this case the edges alone are not enough for determining the solid.



"The data which it is proper to select for determining a solid, are those which leave no indeterminateness, and give absolutely only one solution. And first, the base ABCDE will be determined by this among other modes; by knowing the side AB with the adjacent angles BAC, ABC for the point C; the angles BAD, ABD for the point D; and so for all the rest. Next, let M be a point without the base whose position it is required to determine: this point will be determined, if, imagining the pyramid MABC or simply the plane MAB, we know the angles MAB, ABM, and the inclination of the plane MAB to the base ABC. If by means of three analogous data, the position of each vertex lying without the base of the polyedron is determined, the polyedron, it is evident, will be absolutely determined, and so that two polyedrons constructed with the same data must of necessity be equal; or symmetrically equal, if constructed on different sides of the plane of the base.

"It is not always required to have three data for determining each vertex of a polyedron; for if the point M must be found in a plane already determined, whose intersection with the base is FG, it will be sufficient, after having assumed FG at will, if we know the angles MGF, MFG; and thus one datum less will be enough. If the point M must be found in two planes already determined, or in their common intersection MK, which meets ABC in K, we shall in this case already know the side AK, the angle AKM, and the inclination of the plane AKM to the base; hence it will be enough to have for a new datum the angle MAK. By such means, the number of data necessary for determining a polyedron absolutely and without any ambiguity will always be reduced to A, the number of its edges.

"The side AB and a number A − 1 of given angles determine a polyedron; another side assumed at pleasure and the same angles dertermined a similar polyedron. Hence it follows that *the number of conditions necessary for determining the similarity of two polyedrons belonging to the same species is equal to the number of edges* minus *one*.

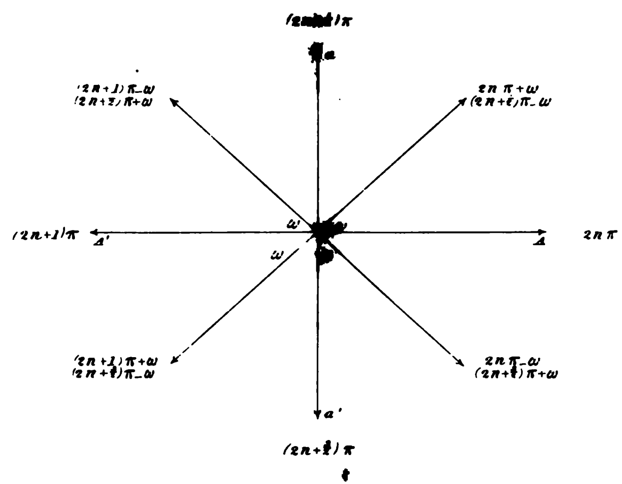
"The question we have just resolved would be much simpler, if, instead of knowing the species of the polyedron, we knew only S the number of its solid angles. In that case, determine three vertices at pleasure by means of a triangle in which are three data; this triangle will be regarded as the base of the solid; then the number of vertices out of this base will be $S - 3$; and since the determination of each of them requires three data, the total number of data necessary for determining the polyedron will evidently be $3 + 3(S - 3)$, or $3S - 6$.

"Hence $3S - 7$ conditions will be necessary for determining the similarity of two polygons having the same number S of solid angles."

THE END.

LONDON:
PRINTED BY THOMAS DAVISON, WHITEFRIARS.

Tab.1.



The most useful formulæ in the

1.* $\sin.^2\omega + \cos.^2\omega = 1.$

2.* $\dfrac{\sin.\omega}{\cos.\omega} = \tan.\omega.$

3.* $\dfrac{\cos.\omega}{\sin.\omega} = \cot.\omega.$

4.* $\tan.\omega \cot.\omega = 1.$

5.* $\sec.\omega \cos.\omega = 1.$

6.* $\operatorname{cosec}.\omega \sin.\omega = 1.$

7.* $1 + \tan.^2\omega = \sec.^2\omega.$

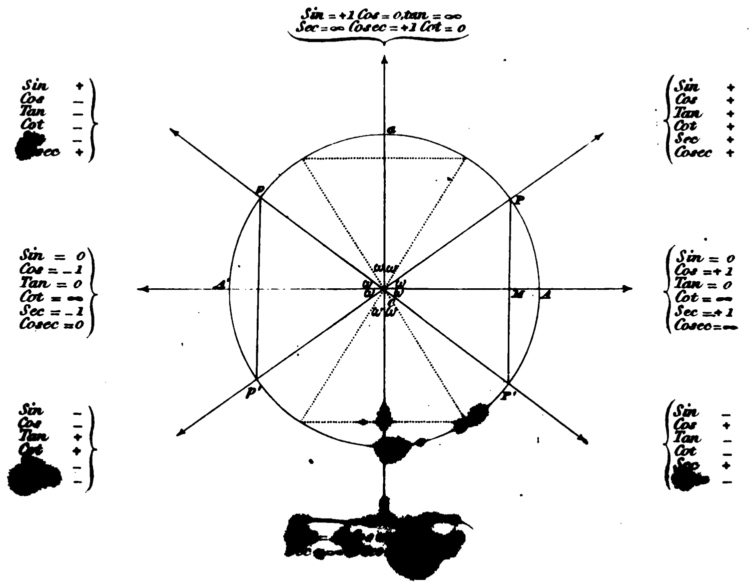
8. $1 + \cot.^2\omega = \operatorname{cosec}.^2\omega.$

9.* $\operatorname{ver.sin}.\omega = 1 - \cos.\omega.$

10. $\operatorname{cover.sin}.\omega = 1 - \sin.\omega.$

11. $\operatorname{suver.sin}.\omega = 1 + \cos.\omega.$

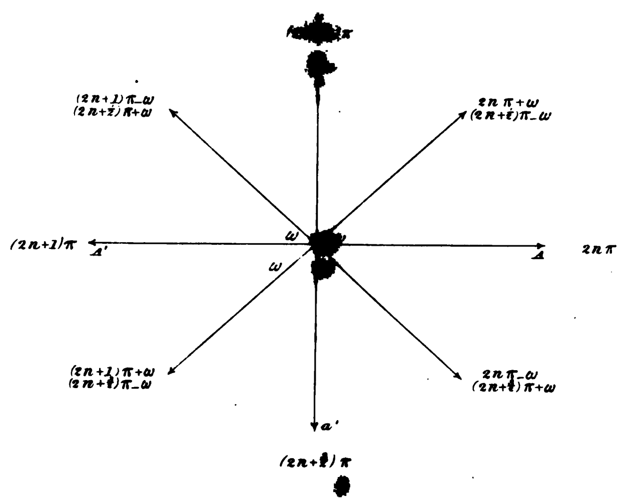
Tab. III.



Sin = +1 Cos = 0, tan = ∞
Sec = ∞ Cosec = +1 Cot = 0
Sin + Cos – Tan – Cot – Sec – Cosec +
Sin + Cos + Tan + Cot + Sec + Cosec +
Sin = 0 Cos = –1 Tan = 0 Cot = ∞ Sec = –1 Cosec = 0
Sin = 0 Cos = +1 Tan = 0 Cot = ∞ Sec = +1 Cosec = ∞
Sin – Cos – Tan + Cot +
Sin – Cos + Tan – Cot – Sec + –
a
P
p
p'
P'
A'
M
A

LONDON:
PRINTED BY THOMAS DAVISON, WHITEFRIARS.

Tab. 1.



$(2n+1)\pi - \omega$
$(2n+\frac{?}{2})\pi + \omega$
$2n\pi + \omega$
$(2n+\frac{?}{2})\pi - \omega$
$(2n+1)\pi$
A'
ω
ω
A
$2n\pi$
$(2n+1)\pi + \omega$
$(2n+\frac{?}{2})\pi - \omega$
$2n\pi - \omega$
$(2n+\frac{?}{2})\pi + \omega$
a'
$(2n+\frac{?}{2})\pi$

The most useful formulæ in the

1.* $\sin.^2\omega + \cos.^2\omega = 1.$

2.* $\dfrac{\sin.\omega}{\cos.\omega} = \tan.\omega.$

3.* $\dfrac{\cos.\omega}{\sin.\omega} = \cot.\omega.$

4.* $\tan.\omega \cot.\omega = 1.$

5.* $\sec.\omega \cos.\omega = 1.$

6.* $\operatorname{cosec}.\omega \sin.\omega = 1.$

7.* $1 + \tan.^2\omega = \sec.^2\omega.$

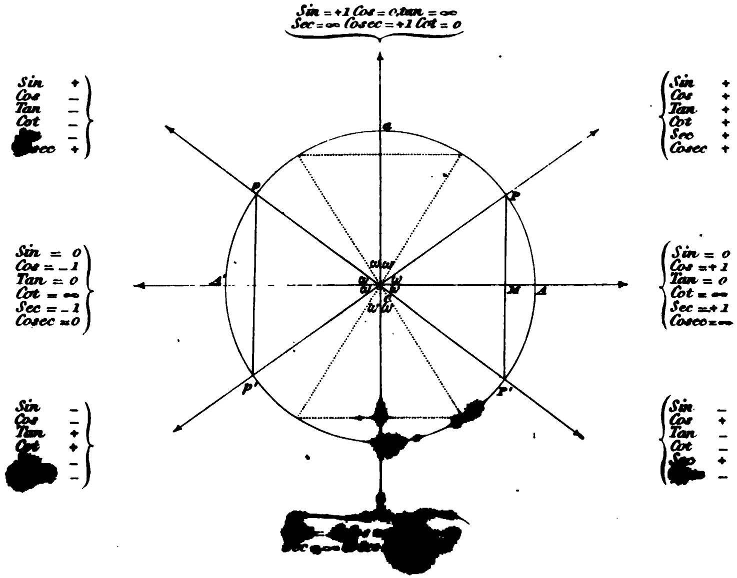
8. $1 + \cot.^2\omega = \operatorname{cosec}.^2\omega.$

9.* $\operatorname{ver.sin}.\omega = 1 - \cos.\omega.$

10. $\operatorname{cover.sin}.\omega = 1 - \sin.\omega.$

11. $\operatorname{suver.sin}.\omega = 1 + \cos.\omega.$

Tab. III.


Sin = +1 Cos = 0, tan = ∞
Sec = ∞ Cosec = +1 Cot = 0
Sin + Cos + Tan + Cot + Sec + Cosec +
Sin = 0 Cos = +1 Tan = 0 Cot = ∞ Sec = +1 Cosec = ∞
Sin = 0 Cos = −1 Tan = 0 Cot = ∞ Sec = −1 Cosec = 0
Sin + Cos − Tan − Cot − Cosec +
Sin − Cos + Tan − Cot −
Sin − Cos − Tan + Cot +